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CATALYTIC BRANCHING RANDOM WALK ON THREE-DIMENSIONAL LATTICE

We consider a critical catalytic continuous time branching random walk on the integer lattice under the assumption that the birth and the death of particles occur at a single source of branching located at the origin. For the introduced joint generating function of the number of particles, the differential and integral equations are obtained in case of $\mathbb{Z}^d$ with arbitrary $d \in \mathbb{N}$. The limit value of the population survival probability as $t \to \infty$ is found as a function of the starting point $x \in \mathbb{Z}^3$. We establish the asymptotic behavior of the probability that the number of particles at the origin at time t is positive. The Yaglom-type conditional limit theorem for the number of particles at the origin is proved. A joint conditional limit distribution of the number of particles at the source and the number of particles outside of it with finite lifetime is studied as well.

1. Introduction

The models described in terms of both the walking and the branching of particles on integer lattices of various dimensions were recently applied in statistical physics and chemistry (see, e.g., [1] and [2]). A type of model depends on certain parameters including the lattice dimension and the number of the sources of birth and death located at lattice points. It is interesting to note that nontrivial effects appear even in the models with a single source of branching (see [3] – [15]). Therefore, we focus on this case permitting to investigate how the lattice dimension influences the asymptotic properties of the system of particles.

An essential role in the following analysis belongs to a continuous time symmetric branching random walk (SBRW) on $\mathbb{Z}^d, d \in \mathbb{N}$. Such a model goes back at least to [3]. The characteristic features of the model are a symmetry and a spatial homogeneity of the random walk transition intensities $a(u, v)$, $u, v \in \mathbb{Z}^d$. The former is equivalent to the self-adjointness of the corresponding infinitesimal transition matrix $A = (a(u, v))_{u,\, v \in \mathbb{Z}^d}$. According to [4] – [9], the asymptotic behavior of the total number of particles and that of the number of particles at the source are determined by the value of parameter $\beta = f'(1)$, where $f(x)$, $x \in [0, 1]$, is the infinitesimal generating function governing the branching mechanism at a point of the birth and the death of particles. Note that the critical value β_c depends mainly on the dimension of the lattice. In particular, under the condition $\beta \geq \beta_c$, the exponential growth of both the total number and the local number of particles occurs. The detailed analysis of the model is developed in monograph [10].

Another model, called a catalytic branching random walk (CBRW), of critical continuous time branching random walk on $\mathbb{Z}^d, d \in \mathbb{N}$, with a single source was proposed for $d = 1$ in [11]. It differs from SBRW by the parameter α controlling the relation of the branching and the walk at the source. The introduction of the parameter α causes an asymmetry of the random walk generator $\bar{A}$. This entails new difficulties for the investigation of the model. Papers [12, 13] for $d = 1$ and [14] for $d = 2$ are devoted to

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the study of CBRW by methods of branching processes. A reformulation of the model description in terms of infinitesimal characteristics was performed in [15]. In this way, the necessary and sufficient conditions of the criticality of CBRW on $\mathbb{Z}^d,\ d \in \mathbb{N}$, were obtained by the methods of differential equations in the Banach spaces and the spectral theory of operators.

The aim of the present paper is to study critical CBRW on a three-dimensional lattice. Seemingly, there are no publications concerning this subject, though such an investigation is of undoubtful interest. As was shown in [10] (p. 34), a random walk with generator $\bar{A}$ on $\mathbb{Z}^d,\ d \in \mathbb{N}$, is transient for $d \geq 3$ and is recurrent for $d = 1$ or $d = 2$. So it is natural to expect new effects in asymptotic properties of CBRW on a three-dimensional lattice which were not observed in the case of lattices of lower dimensions. Indeed, the survival probability of a population of particles on $\mathbb{Z}^3$ has a nonzero limit (as time tends to infinity) in contrast to the case of $\mathbb{Z}$ or $\mathbb{Z}^2$. It is explained by the existence (with nonzero probability) of particles which never reach the source. Rather surprisingly, the asymptotics of the probability of nondegeneracy at the origin for the models of CBRW coincide up to a constant factor on $\mathbb{Z}$ and $\mathbb{Z}^3$ (differing, however, from that in the $\mathbb{Z}^2$ case). Moreover, the corresponding Yaglom-type conditional limit theorems for the generating functions of the number of particles at the source have the same form for $d = 1$ and $d = 3$. Basically, this is due to the coincidence, up to a constant factor, of the asymptotics of the first local moments for the CBRW on $\mathbb{Z}$ and $\mathbb{Z}^3$. It is worth mentioning that we have to consider separately the particles with finite and infinite lifetimes to prove the conditional limit theorem like the second part of Theorem 1 in [11]. One can demonstrate the above-mentioned properties of CBRW on the lattices of different dimensions using the results of [11, 12, 14] and the theorems below. Furthermore, the new differential and integral equations for the introduced joint generating functions of the numbers of particles for the CBRW model on $\mathbb{Z}^d,\ d \in \mathbb{N}$, will be derived while proving these theorems. Some useful auxiliary statements will be provided as well.

Before formulating the main results, we describe the model of CRBW on $\mathbb{Z}^3$ and introduce some necessary notation.

The population of particles is initiated at the time $t = 0$ by a parent particle located at a point $x \in \mathbb{Z}^3$. If $x \neq 0$, then the particle performs a continuous time random walk until it gets to the origin. The random walk outside the origin is specified by an infinitesimal transition matrix $A = (a(u, v))_{u \in \mathbb{Z}^3 \setminus \{0\},\, v \in \mathbb{Z}^3}$ and is assumed to be symmetric, homogeneous, irreducible, and having zero mean and a finite variance of jumps, that is,

$$a(u, v) = a(v, u),\ \ a(u, v) = a(0, v - u) \stackrel{def}{=} a(v - u)\ \text{ with }\ a(v) \geq 0,\ v \neq 0,\ \ a(0) < 0,$$

$$\sum_{v \in \mathbb{Z}^3} a(v) = 0 \ \text{ and } \sum_{v \in \mathbb{Z}^3} \|v\|^2 a(v) < \infty.$$

If $x = 0$ or if a particle hits the origin, it spends an exponentially distributed time with parameter 1 at the source of branching. Then the particle either dies with the probability α, having produced just before the death a random number of offsprings ξ, or leaves the source of branching with the probability $1 - \alpha$. The branching of the particle is governed by the offspring generating function

$$\mathcal{F}(s) \stackrel{def}{=} \mathsf{E} s^{\xi} = \sum\nolimits_{k=0}^{\infty} f_k s^k,$$

where $f_k,\ k > 0$, is the probability to produce k offsprings, and f_0 is a probability to die. We suppose that $\mathcal{F}'(1) = 1 + \bar{\beta}_c\, \alpha^{-1}$, where the explicit formula for $\bar{\beta}_c$ is given below in (2), and, moreover, we assume $\sigma^2 = \mathcal{F}''(1) < \infty$. Further, we will clarify why it is natural to call such a process the critical one. The probability of the transition from the origin to a point $y \neq 0$ equals $a(0, y) = -(1 - \alpha) a(y) a^{-1}(0)$. The new particles behave independently and stochastically in the same way as the parent ones.

In what follows, we often assume that the starting point is $x = 0$.

According to the notation of [15], we will write

$$\bar{f}(s) = \alpha \left(\sum_{k \neq 1} f_k s^k + (f_1 - 1)s \right) = \alpha(\mathcal{F}(s) - s).$$

Let $\bar{p}(t, x, y)$ be the transition probability of a random walk for the CBRW model, that is, the probability that, after starting at the time $t = 0$ at a point x, the particle is located at a point y at a time t. We now introduce the Laplace transform of the transition probability

$$\overline{G}_\lambda(x, y) = \int_0^\infty e^{-\lambda t} \bar{p}(t, x, y)\, dt, \quad x, y \in \mathbb{Z}^3,\ \lambda \geq 0.$$

By $\zeta(t)$, we denote the number of particles at the origin and write $\mu(t)$ for the number of particles outside the origin and $\eta(t) = \zeta(t) + \mu(t)$ for the total number of particles on the lattice at a time t.

The following theorem establishes the limit behavior for the generating function

$$F(s, t, x) = \mathsf{E}_x s^{\eta(t)}$$

of the total number of particles and for the survival probability of the population of particles $Q(t, x) = 1 - F(0, t, x)$, provided that the process starts at x.

Theorem 1.1. *For any $s \in [0, 1]$ and $x \in \mathbb{Z}^3$, one has*

$$\lim_{t \to \infty} F(s, t, x) = 1 - c_3(s, x), \quad \lim_{t \to \infty} Q(t, x) = c_3(x),$$

where $c_3(x) = c_3(0, x) > 0$, and $c_3(s, x)$ is the unique root of the equation

$$\frac{1 - c_3(s, x) - s}{\overline{G}_0(x, 0)} = \bar{f}(1 - c_3(s, 0)). \tag{1}$$

Corollary 1.1. *Let $s \in [0, 1]$ and $x \in \mathbb{Z}^3$. There exists*

$$\lim_{t \to \infty} \mathsf{E}_x \{ s^{\eta(t)} | \eta(t) > 0 \} = \frac{(1 - \bar{\beta}_c \overline{G}_0(x, 0))s + \bar{\beta}_c \overline{G}_0(x, 0)(c_3(0) - c_3(s, 0))}{1 - \bar{\beta}_c \overline{G}_0(x, 0)(1 - c_3(0))},$$

where

$$\bar{\beta}_c = \frac{1}{\overline{G}_0(0, 0)} > 0. \tag{2}$$

The asymptotics of the probability $q(t) = \mathsf{P}_0(\zeta(t) > 0)$ of nondegeneracy at the origin is given by

Theorem 1.2. *As $t \to \infty$, the following relation is valid:*

$$q(t) \sim \frac{2\gamma_3 \sqrt{\pi}\, a(0)}{\alpha(\alpha - 1)\, \sigma^2\, \overline{G}_0^{\,2}(0, 0)} \, \frac{1}{\sqrt{t} \ln t}.$$

Here, γ_3 is a positive constant calculated in [10], *p.* 31.

The next theorem describes the limit behavior of properly normed $\zeta(t)$ given $\zeta(t) > 0$.

Theorem 1.3. *For every $\lambda \in [0, \infty)$, one has*

$$\lim_{t \to \infty} \mathsf{E}_0 \left(\exp \left\{ - \frac{\lambda \zeta(t)}{\mathsf{E}_0(\zeta(t) | \zeta(t) > 0)} \right\} \middle| \zeta(t) > 0 \right) = \frac{1}{\lambda + 1}.$$

Note that the last expression is the Laplace transform of the exponential distribution with parameter 1.

Since the random walk on $\mathbb{Z}^3$ under consideration is transient, it is natural for the particles outside the origin at a time t to distinguish between those of a finite lifetime (they will return to the source eventually) and the infinite lifetime (they will walk on $\mathbb{Z}^3\backslash 0$). Let $\mu_0(t)$ and $\mu_1(t)$ be the number of particles outside the origin of a finite lifetime and of the infinite lifetime, respectively, at the time t. Hence, $\eta_0(t) \stackrel{def}{=} \zeta(t) + \mu_0(t)$ is the total number of particles with a finite lifetime.

Theorem 1.4. *If $s_1, s_2 \in [0,1]$, then*

$$\lim_{t\to\infty} \mathsf{E}_0(s_1^{\zeta(t)}\, s_2^{\mu_0(t)} | \eta_0(t) > 0) = 1 - \sqrt{1-s_2}.$$

Remark 1.1. These results imply that, in contrast to the model where $\mathcal{F}'(1) > 1 + \bar{\beta}_c\alpha^{-1}$ (see [15]), the total number of particles does not grow exponentially when $\mathcal{F}'(1) = 1 + \bar{\beta}_c\alpha^{-1}$. This observation justifies the term critical branching random walk.

2. Auxiliary Results

In view of the formulas for the CBRW transition probabilities (see [15]), the following backward Kolmogorov equations for $\bar{p}\,(t,x,y)$ hold:

$$\partial_t \bar{p}(t,x,y) = (\bar{A}\bar{p}(t,\cdot,y))(x), \quad \bar{p}(0,x,y) = \delta_y(x), \quad x,y \in \mathbb{Z}^3. \tag{3}$$

Here, the matrix

$$\bar{A} = (\bar{a}(u,v))_{u,v\in\mathbb{Z}^3} = \left(I + \left(\frac{\alpha-1}{a(0)} - 1\right)\delta_0\delta_0^T\right)A$$

specifies a linear bounded operator in the Banach space $l_\infty(\mathbb{Z}^3)$. Its action is given by the rule

$$(\bar{A}\,r(\cdot))(u) = \sum\nolimits_{v\in\mathbb{Z}^3} \bar{a}(u,v)r(v), \quad r(\cdot) \in l_\infty(\mathbb{Z}^3).$$

As usual, I is the identity operator in the mentioned space, δ_0 is the column vector such that $\delta_0(0) = 1$ and $\delta_0(x) = 0$ for $x \neq 0$, whereas T denotes the transposition.

We consider Eq. (3) as an inhomogeneous one for the differential equation

$$\partial_t p(t,x,y) = (Ap(t,\cdot,y))(x), \quad p(0,x,y) = \delta_y(x), \quad x,y \in \mathbb{Z}^3. \tag{4}$$

The solution of the last equation is studied in [10] in detail. In particular, as was shown there, the SBRW transition probabilities $p(t,x,y)$ for $x, y \in \mathbb{Z}^3$ possess the property

$$G_\lambda(x,y) \stackrel{def}{=} \int_0^\infty e^{-\lambda t} p(t,x,y)\,dt < \infty, \quad \lambda \geq 0, \quad x,y \in \mathbb{Z}^3. \tag{5}$$

Lemma 2.1. *The following equality is valid:*

$$\overline{G}_0(0,0) = \frac{a(0)}{\alpha-1} G_0(0,0).$$

Proof. Apply the constant variation formula to Eqs. (3) and (4). Then, for $x = y = 0$, we obtain

$$\bar{p}(t,0,0) = p(t,0,0) + \left(1 - \frac{a(0)}{\alpha-1}\right)\int_0^t p(t-u,0,0)\partial_u \bar{p}\,(u,0,0)\,du. \tag{6}$$

Since

$$\int_0^\infty e^{-\lambda t}\bar{p}\,'(t,0,0)\,dt = \lambda\overline{G}_\lambda(0,0) - 1,$$

implementing the Laplace transformation of both sides of (6) yields

$$\overline{G}_\lambda(0,0) = \frac{a(0)\,G_\lambda(0,0)}{\alpha - 1 - \lambda(\alpha-1-a(0))G_\lambda(0,0)}.$$

Setting here $\lambda = 0$ proves the claim. $\square$

Due to [15] and Lemma 2.1, one has $\bar{\beta}_c = 1/\overline{G}_0(0,0)$. Moreover, relation (5) and Lemma 2.1 entail the inequality $\overline{G}_0(0,0) < \infty$ that amounts to $\bar{\beta}_c > 0$.

For a function $\chi(t),\ t \geq 0$, we set

$$\hat{\chi}(\lambda) \stackrel{def}{=} \int_0^\infty e^{-\lambda t}\chi(t)\,dt, \quad \lambda \geq 0.$$

Moreover, if $\chi(t)$ is nonnegative and nondecreasing, then

$$\check{\chi}(\lambda) \stackrel{def}{=} \int_0^\infty e^{-\lambda t}\,d\chi(t), \quad \lambda \geq 0.$$

We temporarily forget that there is the source of branching on the lattice and consider an ordinary random walk on $\mathbb{Z}^3$ with generator $\bar{A}$. Denote, by τ_1, the time spent by a particle at the origin until it leaves the origin. Then

$$G_1(t) \stackrel{def}{=} \mathsf{P}_0(\tau_1 \leq t) = 1 - e^{-(1-\alpha)t}.$$

Let τ_2 be the time spent by this particle outside the origin until its first return to the origin. The corresponding distribution function is defined by way of

$$G_2(t,x) \stackrel{def}{=} \mathsf{P}_x(\tau_2 \leq t).$$

Here, $x \in \mathbb{Z}^3$ indicates the starting point of the random walk.

Assume that the random walk starts at a point $x \in \mathbb{Z}^3$ at the initial time $t = 0$. In what follows, we will be interested in the probability of the event that the particle will never hit the origin. It is defined to be

$$q(x) \stackrel{def}{=} \lim_{t\to\infty} (1 - G_2(t,x)).$$

Lemma 2.2. *If $x \neq 0$, then*

$$q(x) = 1 - \frac{\overline{G}_0(x,0)}{\overline{G}_0(0,0)} = 1 - \bar{\beta}_c\overline{G}_0(x,0).$$

Proof. Using the continuous analog of the total probability formula, one has

$$\bar{p}(t,x,0) = \int_0^t \bar{p}(t-u,0,0)\,d(G_2(x,\cdot) * G_1)(u),$$

where $*$ denotes the convolution. Apply the Laplace transformation to both sides of the previous equation. In view of the relation

$$\check{G}_1(\lambda) = \frac{1-\alpha}{\lambda + 1 - \alpha},$$

we get

$$\check{G}_2(x,\lambda) = \frac{\overline{G}_\lambda(x,0)(\lambda+1-\alpha)}{\overline{G}_\lambda(0,0)(1-\alpha)} \sim \frac{\overline{G}_0(x,0)}{\overline{G}_0(0,0)}, \quad \lambda \to 0+.$$

Since $\lim\limits_{\lambda\to 0} \int_0^\infty e^{-\lambda t}\,dG_2(t,x) = \int_0^\infty dG_2(t,x) = \lim\limits_{t\to\infty} G_2(t,x)$, we complete the proof of the lemma. □

Return to the CBRW model. The rest of the section is devoted to the investigation of the first local moments $\overline{m}_1(t,x,0) \stackrel{def}{=} \mathsf{E}_x\zeta(t),\ t \geq 0,\ x \in \mathbb{Z}^3$. In particular, we dwell on the asymptotic behavior of the function $\overline{m}_1(t) \stackrel{def}{=} \overline{m}_1(t,0,0)$.

We will often use the properties of the first local moments $m_1(t,x,0),\ t \geq 0,\ x \in \mathbb{Z}^3$, introduced for SBRW model. These properties are described in [10] in detail. For brevity, we will write $m_1(t) \stackrel{def}{=} m_1(t,0,0)$,

$$\overline{M}_\lambda(x,y) \stackrel{def}{=} \int_0^\infty e^{-\lambda t}\overline{m}_1(t,x,y)\,dt, \quad M_\lambda(x,y) \stackrel{def}{=} \int_0^\infty e^{-\lambda t}m_1(t,x,y)\,dt.$$

Lemma 2.3. *The function $\overline{m}_1(t)$ is monotone decreasing, and, as $t \to \infty$, one has*

$$\overline{m}_1(t) \sim \frac{(\alpha-1)\overline{G}_0^2(0,0)}{a(0)\,\gamma_3\sqrt{\pi}\sqrt{t}}.$$

Proof. As was shown in [15], the first moment of the number of particles at the source satisfies the differential equation

$$\partial_t \overline{m}_1(t,x,0) = (\bar{A}\overline{m}_1(t,\cdot,0))(x) + \bar{\beta}_c(\varDelta_0 \overline{m}_1(t,\cdot,0))(x) \tag{7}$$

with the initial condition $\overline{m}_1(0,x,0) = \delta_0(x)$. Here, the linear bounded operator $\varDelta_0 \overset{def}{=} \delta_0\delta_0^T$ acts in the space $l_\infty(\mathbb{Z}^3)$. Consider the above equation as inhomogeneous for the differential equation (see [10], Theorem 1.3.1.)

$$\partial m_1(t,x,0) = (Am_1(t,\cdot,0))(x) + \beta_c(\varDelta_0 m_1(t,\cdot,0))(x) \tag{8}$$

with the initial condition $m_1(0,x,0) = \delta_0(x)$. The constant variation formula combined with Lemma 2.1, the equality $\beta_c = (G_0(0,0))^{-1}$ (see [10]), and the relation

$$(\bar{A}\overline{m}_1(t,\cdot,0))(0) = \frac{a(0)}{\alpha-1}(\partial_t \overline{m}_1(t,0,0) - \bar{\beta}_c \overline{m}_1(t,0,0))$$

yields

$$\overline{m}_1(t,x,0) = m_1(t,x,0) + \left(1 - \frac{a(0)}{\alpha-1}\right)\int_0^t m_1(t-u,x,0)\,\partial_u \overline{m}_1(u,0,0)\,du. \tag{9}$$

E.B. Yarovaya established the monotonicity of $\overline{m}_1(t)$ by reducing the asymmetric operator $\bar{A}$ to a self-adjoint one (see [16]) and then using the same methods as were employed to prove the monotonicity of $m_1(t)$ in Lemma 3.3.5 in [10]. Thus, we turn to determining the asymptotics of $\overline{m}_1(t)$ as $t \to \infty$.

We now perform the Laplace–Stieltjes transformation of both sides of (9) when $x = 0$. Since

$$\int_0^\infty e^{-\lambda t} d\,\overline{m}_1(t) = \lambda \overline{M}_\lambda(0,0) - 1,$$

we get

$$\overline{M}_\lambda(0,0) = \frac{a(0)\,M_\lambda(0,0)}{\alpha - 1 - (\alpha-1-a(0))\,\lambda M_\lambda(0,0)}. \tag{10}$$

The asymptotic behavior of $m_1(t)$ found in [10] is as follows:

$$m_1(t) \sim \frac{G_0^2(0,0)}{\gamma_3\sqrt{\pi}\sqrt{t}}, \quad t \to \infty. \tag{11}$$

Hence, the application of a Tauberian theorem (Theorem 4, Ch. XIII, §5, [17]) implies

$$M_\lambda(0,0) \sim \frac{G_0^2(0,0)}{\gamma_3\sqrt{\lambda}}, \quad \lambda \to 0+.$$

On account of (10) and Lemma 2.1, we obtain the asymptotics

$$\overline{M}_\lambda(0,0) \sim \frac{(\alpha-1)\,\overline{G}_0^2(0,0)}{a(0)\,\gamma_3\sqrt{\lambda}}, \quad \lambda \to 0+. \tag{12}$$

Using the mentioned Tauberian theorem and the monotonicity of $\overline{m}_1(t)$, $t \geq 0$, leads to the second part of the required statement. □

3. The Total Number of Particles

In this section, we will prove Theorem 1.1 and derive its Corollary 1.1.

By Theorem 1 in [15], the generating function $F(s,t,x), s\in[0,1], t\in[0,\infty), x\in\mathbb{Z}^3$, satisfies the differential equation in the Banach space $l_\infty(\mathbb{Z}^3)$

$$\partial_t F(s,t,x) = (\bar{A}F(s,t,\cdot))(x) + (\Delta_0 \bar{f}(F(s,t,\cdot)))(x) \tag{13}$$

with the initial condition $F(s,0,x) = s$. Taking (3) into account and employing the constant variation formula (see [18], p. 105), we infer from the last equation that

$$F(s,t,x) = s + \int_0^t \bar{p}(t-u,x,0)\bar{f}(F(s,u,0))\,du. \tag{14}$$

The function $F(s,t,x)$ is monotone in the variable t, when s and x are fixed. This claim is based on the existence of positive solutions of differential equations in $l_\infty(\mathbb{Z}^d)$ with the off-diagonal positive right-hand side. In its turn, the proof of the last fact is a counterpart of that for finite-dimensional spaces (for the latter, see [19], p. 63).

The monotonicity in t and the boundedness of $F(s,t,x)$ entail the existence of a limit $\lim_{t\to\infty} F(s,t,x) = 1 - c_3(s,x)$, when $s\in[0,1]$ and $x\in\mathbb{Z}^3$ are fixed. Let us determine the limit function $c_3(s,x)$. We let t tend to infinity in (14) and use the lemma on asymptotic properties of integral convolutions of functions of the power-logarithmic kind (see Lemma 5.1.2. in [10]) to derive the equation

$$1 - c_3(s,x) = s + \overline{G}_0(x,0)\bar{f}(1 - c_3(s,0)) \tag{15}$$

which is equivalent to (1). For $x = 0$, the last equation can be rewritten as follows:

$$(1-s)\bar{\beta}_c = \bar{f}(1 - c_3(s,0)) + \bar{\beta}_c\, c_3(s,0). \tag{16}$$

The continuous function $\bar{f}(1-y) + \bar{\beta}_c y$ is increasing on $[0,1]$ and varies from 0 to the value greater than $\bar{\beta}_c$. One has $(1-s)\bar{\beta}_c \le \bar{\beta}_c$ for every $s\in[0,1]$. Hence, there exists the unique root $c_3(s,0)$ of Eq. (16). Therefore, we can claim the existence and uniqueness of $c_3(s,x)$ satisfying (15).

To complete the proof of Theorem 1.1, we will demonstrate the positivity of $c_3(x) = c_3(0,x)$, $x\in\mathbb{Z}^3$. Indeed, assuming that $c_3(0) = 0$, we arrive at a contradiction, as it does not verify (1) for $x = s = 0$. Hence, $c_3(0) > 0$. If $x \ne 0$, then the desired fact ensues from the inequalities

$$\overline{G}_0(x,0) = \frac{a(0)}{\alpha - 1} G_0(x,0) \le \frac{a(0)}{\alpha-1} G_0(0,0) = \overline{G}_0(0,0),$$

$$c_3(x) = 1 - \overline{G}_0(x,0)\bar{f}(1 - c_3(0)) \ge 1 - \overline{G}_0(0,0)\bar{f}(1 - c_3(0)) = c_3(0) > 0.$$

Note that the first of them is a consequence of expression (2.2.2) in [10].

The proof of Theorem 1.1 is complete. $\square$

In view of Theorem 1.1, one has

$$\lim_{t\to\infty} \mathsf{E}_x(s^{\eta(t)} | \eta(t) > 0) = \lim_{t\to\infty} \frac{F(s,t,x) - (1 - Q(t,x))}{Q(t,x)} = \frac{c_3(x) - c_3(s,x)}{c_3(x)}. \tag{17}$$

Since $\bar{\beta}_c = (\overline{G}_0(0,0))^{-1}$, Eq. (1) guarantees that

$$c_3(s,x) = 1 - s(1 - \bar{\beta}_c\overline{G}_0(x,0)) - \bar{\beta}_c\overline{G}_0(x,0)(1 - c_3(s,0)).$$

The corollary is now evident due to the explicit expression for $c_3(s,x)$ and (17). $\square$

4. Equations for Joint Generating Functions

Define the joint generating function for three random variables, being the number of particles at the origin, outside the origin with a finite lifetime and infinite lifetime, respectively, by way of

$$F(t,s_1,s_2,s_3,x) = \mathbf{E}_x s_1^{\zeta(t)} s_2^{\mu_0(t)} s_3^{\mu_1(t)}, \quad s_1,s_2,s_3\in[0,1], \quad t \ge 0, \quad x\in\mathbb{Z}^3.$$

Taking the particle evolution during the time interval $[0,h]$ into account and using a Markov property of the random walk and the branching as well, we set up the differential equation for the introduced function (compare with Lemma 1.2.1 in [10])

$$\partial_t F(t,s_1,s_2,s_3,x) = (\bar{A}F(t,s_1,s_2,s_3,\cdot))(x) + (\varDelta_0 \bar{f}(F(t,s_1,s_2,s_3,\cdot)))(x). \tag{18}$$

Starting from a point $x \neq 0$ at the initial time $t=0$, the particle has the infinite lifetime with the probability $q(x)$ and has a finite lifetime with the probability $1-q(x)$. Hence, the initial condition for (18) is as follows:

$$F(0,s_1,s_2,s_3,x) = \delta_0(x)(s_1-s_2) + s_2 + q(x)(s_3-s_2)(1-\delta_0(x)). \tag{19}$$

Introduce an auxiliary function

$$F_1(t,x) \overset{def}{=} F(t,s_1,s_2,s_3,x) - s_2 - q(x)(s_3-s_2)(1-\delta_0(x)).$$

For brevity, we will omit some of its arguments. Note that

$$\partial_t F_1(t,x) = \partial_t F(t,s_1,s_2,s_3,x),$$

$$(\bar{A}F_1(t,\cdot))(x) = (\bar{A}F(t,s_1,s_2,s_3,\cdot))(x) - \delta_0(x)\bar{\beta}_c(s_3-s_2),$$

because combining the formulas for $q(x)$ and for $\overline{G}_0(x,0)$ with equation (3) shows that the following equality is valid:

$$\begin{aligned}(\bar{A}\,q(\cdot)(1-\delta_0(\cdot)))(x) &= -\bar{\beta}_c \sum_{x'\in\mathbb{Z}^3} \bar{a}(x,x')\,\overline{G}_0(x',0) \\ &= -\bar{\beta}_c \int_0^\infty (\bar{A}\bar{p}(t,\cdot,0))(x)\,dt = -\bar{\beta}_c\int_0^\infty d\bar{p}(t,x,0).\end{aligned}$$

Thus, $F_1(t,x)$ satisfies the differential equation

$$\begin{aligned}\partial_t F_1(t,x) &= (\bar{A}F_1(t,\cdot))(x) \\ &+ (\varDelta_0[\bar{f}(F_1(t,\cdot)+s_2+q(\cdot)(s_3-s_2)(1-\delta_0(\cdot))) + \bar{\beta}_c(s_3-s_2)])(x)\end{aligned}$$

with the initial condition $F_1(t,x) = \delta_0(x)(s_1-s_2)$.

Consider the last equation as an inhomogeneous one for the differential equation

$$\partial_t M_1(t,x) = (\bar{A}M_1(t,\cdot))(x) + \bar{\beta}_c(\varDelta_0 M_1(t,\cdot))(x)$$

with the initial condition $M_1(0,x) = \delta_0(x)(s_1-s_2)$. The solution of the above equation is $M_1(t,x) = (s_1-s_2)\,\overline{m}_1(t,x,0)$. Applying the constant variation formula gives us the integral equation

$$\begin{aligned}F_1(t,x) &= (s_1-s_2)\,\overline{m}_1(t,x,0) \\ &+ \int_0^t \overline{m}_1(t-u,x,0)[\bar{f}(F_1(u,0)+s_2) + \bar{\beta}_c(s_3-s_2) - \bar{\beta}_c F_1(u,0)]\,du.\end{aligned}$$

Returning to the initial notation, we get

$$\begin{aligned}F(t,s_1,s_2,s_3,x) &= s_2 + q(x)(s_3-s_2)(1-\delta_0(x)) + (s_1-s_2)\overline{m}_1(t,x,0) \\ &+ \int_0^t \overline{m}_1(t-u,x,0)[\bar{f}(F(u,s_1,s_2,s_3,0)) + \bar{\beta}_c s_3 - \bar{\beta}_c F(u,s_1,s_2,s_3,0)]\,du.\end{aligned} \tag{20}$$

It is possible to introduce a joint generating function of another kind. There are no difficulties in deriving integral equations for other joint generating functions from Eq. (20). Further, we will need some of them. In particular, we set $q(t,s) \overset{def}{=} 1 - \mathsf{E}_0 s^{\zeta(t)}$. Substituting $s_1 = s$ and $s_2 = s_3 = 1$ in (20) leads to the relation

$$q(t,s) = (1-s)\,\overline{m}_1(t) - \int_0^t \overline{m}_1(t-u)\,h(q(u,s))\,du, \tag{21}$$

where $h(x) \overset{def}{=} \bar{f}(1-x) + \bar{\beta}_c x$.

Setting $s_3 = 1$ in (20) yields

$$Q(t,s_1,s_2) = 1 - s_2 - (s_1 - s_2)\,\overline{m}_1(t) - \int_0^t \overline{m}_1(t-u)h(Q(u,s_1,s_2))\,du. \tag{22}$$

Here, $Q(t,s_1,s_2) \stackrel{def}{=} 1 - \mathsf{E}_0 s_1^{\zeta(t)} s_2^{\mu_0(t)}$.

5. Particles at the Origin

The size of the paper allows us to give only a sketch of the proofs of Theorems 1.2 and 1.3. Substituting $s = 0$ in (21), we get the integral equation for the probability $q(t)$ of nondegeneracy at the origin. The obtained equation is similar to Eq. (22) in [11] for the same probability in the CBRW model in the case of the lattice $\mathbb{Z}$. Furthermore, in view of Lemma 2.3 of the present paper and Lemma 8 in [11], the asymptotic behavior of $\overline{m}_1(t)$ in these models is the same up to a constant factor. Hence, the asymptotics of $q(t)$ in CBRW models on $\mathbb{Z}$ and on $\mathbb{Z}^3$ coincides modulo a constant factor. Thus, the further proof of Theorem 1.2 is performed in the same manner as that of Theorem 2 in [11].

With regard to Theorem 1.3, one has the analogous situation. Moreover, by reason of the coincidence of the asymptotics of the first local moments and due to the similarity of the integral equation (21) and the equation in [12] for the same generating function in the model of CBRW on $\mathbb{Z}$, the corresponding conditional limit theorems have a similar form. So the proof of Theorem 1.3 repeats that of Theorem 4 in [12] almost literally.

6. Joint Conditional Limit Distribution

This section is devoted to the proof of Theorem 1.4. Let us begin with determining the asymptotic behavior of $Q(t,s,s) = 1 - \mathsf{E}_0 s^{\eta_0(t)}$. This function is a solution of the integral equation (22) when $s_1 = s_2 = s$. The argument analogous to that in Section 3 of the present paper leads to the proof of the monotonicity of $Q(t,s,s)$ in the variable t when s is fixed. By virtue of the monotonicity in t and the boundedness of $Q(t,s,s)$, there exists a limit of this function as $t \to \infty$. The limit equals zero since, otherwise, the integral on the right-hand side of (22) diverges. Thereby, $Q(t,s,s) = o(1),\ t \to \infty,\ s \in [0,1]$. Consequently, we have

$$\widehat{Q}(\lambda,s,s) = o(\lambda^{-1}), \quad \lambda \to 0+\,.$$

On account of Lemma 2.3, applying the Laplace transformation to both sides of (22) when $s_1 = s_2 = s$ gives

$$\widehat{h}(\lambda,s,s) \sim \frac{1-s}{\lambda \overline{M}_\lambda(0,0)} \sim \frac{(1-s)\,a(0)\,\gamma_3}{\sqrt{\lambda}(\alpha-1)\,\overline{G}_0^{\,2}(0,0)}, \quad \lambda \to 0+\,.$$

Therefore, by a Tauberian theorem (Theorem 4, Ch. XIII, §5, [17]), we get

$$h(Q(t,s,s)) \sim \frac{(1-s)\,a(0)\,\gamma_3}{\sqrt{t}\sqrt{\pi}(\alpha-1)\,\overline{G}_0^{\,2}(0,0)}, \quad t \to \infty.$$

In view of $h(x) \sim \alpha\sigma^2 x^2/2,\ x \to 0+$, one has

$$Q(t,s,s) \sim \sqrt{\frac{2(1-s)\,a(0)\,\gamma_3}{\alpha\sigma^2\sqrt{\pi}(\alpha-1)\overline{G}_0^{\,2}(0,0)}} \cdot \frac{1}{\sqrt[4]{t}}, \quad t \to \infty. \tag{23}$$

Next we show that

$$Q(t,s_1,s_2) - Q(t,s_2,s_2) = o(Q(t,0,0)), \quad t \to \infty. \tag{24}$$

Actually, the above relation follows from Theorem 1.2, formula (23), and the inequality

$$|Q(t,s_1,s_2) - Q(t,s_2,s_2)| = |\mathsf{E}_0 s_2^{\mu_0(t)}(s_1^{\zeta(t)} - s_2^{\zeta(t)})| \le |\mathsf{E}_0(s_1^{\zeta(t)} - s_2^{\zeta(t)})| \le q(t).$$

Combining the relation (24) and the chain of relations

$$\lim_{t\to\infty} \mathsf{E}_0(s_1^{\zeta(t)} s_2^{\mu_0(t)} | \eta_0(t) > 0)$$
$$= \lim_{t\to\infty} \frac{Q(t,0,0) - Q(t,s_1,s_2)}{Q(t,0,0)} = \lim_{t\to\infty} \frac{Q(t,0,0) - Q(t,s_2,s_2)}{Q(t,0,0)}$$

with (23) completes the proof of Theorem 1.4. □

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