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ON THE EXACT ORDER OF GROWTH OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH TIME-DEPENDENT COEFFICIENTS

We study the exact order of growth of the solution of the stochastic differential equation $d\eta(t) = g(\eta(t))\,\varphi(t)dt+\sigma(\eta(t))\,\theta(t)dw(t)$, $X(0) = b$, where w is the standard Wiener process, b is a nonrandom positive constant, g, σ are continuous positive functions, and φ and θ are real continuous functions such that a continuous solution η exists. As an application of these results, we discuss the problem of asymptotic equivalence for solutions of stochastic differential equations.

1. Introduction

Gikhman and Skorokhod [9], Keller *et al.* [10], and later Buldygin *et al.* [1], [5]–[8] considered the exact order of growth of solutions of autonomous stochastic differential equations (SDE's) and found conditions, under which these solutions are asymptotically equivalent, as $t \to \infty$, to solutions of ordinary differential equations (ODE's). The same problem for SDE's with the time-dependent coefficients of drift and diffusion of the form

$$g(t,x) = \varphi(t)g(x), \quad \sigma(t,x) = \theta(t)\sigma(x), \quad t \ge 0, \quad x \in \mathbf{R} = (-\infty,\infty),$$

where g, φ, and σ are positive functions, was considered in paper [2]. Moreover, Buldygin *et al.* [8] gave conditions, under which solutions of two different autonomous SDE's are asymptotically equivalent. In this paper, we continue the similar investigations for SDE's with time-dependent coefficients with alternating signs.

Consider, for $k = 1,2$, the stochastic differential equations

$$d\eta_k(t) = g_k(\eta_k(t))\,\varphi_k(t)dt + \sigma_k(\eta_k(t))\,\theta_k(t)dw_k(t),\ t \ge 0; \tag{1}$$

$$\eta_k(0) \equiv b_k > 0,$$

where w_k, $k = 1,2$, are standard Wiener processes defined on a common probability space; b_k, $k = 1,2$, are nonrandom positive constants; φ_k, θ_k, $k = 1,2$, are real continuous functions, and g_k, σ_k, $k = 1,2$, are positive continuous functions such that, for each $k = 1,2$, SDE (1) has a continuous Itô-solution $\eta_k = (\eta_k(t), t \ge 0)$.

For $k = 1,2$, we denote, by $\mu_k = (\mu_k(t), t \ge 0)$, the continuous solution of the Cauchy problem for the ODE's corresponding to (1) with $\sigma_k \equiv 0$, i.e.

$$d\mu_k(t) = g_k(\mu_k(t))\,\varphi_k(t)dt,\ t \ge 0, \quad \mu_k(0) = b_k > 0 \quad (k = 1,2). \tag{2}$$

We assume that, for each $k = 1,2$, the functions g_k and φ_k are such that the continuous solution μ_k exists and satisfies the relation

$$\lim_{t\to\infty} \mu_k(t) = \infty.$$

Three following problems will be considered.

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The first problem is to study, under which conditions the solutions of SDE's (1) and their corresponding ODE's (2) are asymptotically equivalent almost surely (a.s.) on the set $\{\lim_{t\to\infty} \eta_k(t) = \infty\}$, i.e.

$$\lim_{t\to\infty} \frac{\eta_k(t)}{\mu_k(t)} = 1 \quad \text{a.s. on the set} \quad \left\{\lim_{t\to\infty} \eta_k(t) = \infty\right\}, \quad k = 1, 2.$$

The second problem is to study, under which conditions the solutions of ODE's (2) are asymptotically equivalent, i.e.

$$\lim_{t\to\infty} \frac{\mu_1(t)}{\mu_2(t)} = 1. \tag{3}$$

Finally, the third problem is to consider the conditions, under which solutions of the SDE's (1) are asymptotically equivalent a.s., i.e.

$$\lim_{t\to\infty} \frac{\eta_1(t)}{\eta_2(t)} = 1 \quad \text{a.s. on the set} \quad \left\{\lim_{t\to\infty} \eta_1(t) = \infty\right\} \cap \left\{\lim_{t\to\infty} \eta_2(t) = \infty\right\}. \tag{4}$$

All these problems are closely connected, and the solution of the third problem follows from two first ones.

2. Asymptotic Equivalence of Solutions of SDE's and ODE's

In this section, we consider the exact order of growth of the solution $\eta = (\eta(t), t \geq 0)$ of the SDE

$$d\eta(t) = g\left(\eta(t)\right)\varphi(t)dt + \sigma\left(\eta(t)\right)\theta(t)dw(t), \quad t \geq 0; \tag{5}$$
$$\eta(0) = b > 0,$$

were w is a standard Wiener process, and b is a nonrandom positive constant. We assume that $\varphi = (\varphi(x), x \in \mathbf{R})$ and $\theta = (\theta(x), x \in \mathbf{R})$ are real continuous functions, and $g = (g(x), x \in \mathbf{R})$ and $\sigma = (\sigma(x), x \in \mathbf{R})$ are positive continuous functions such that (5) has continuous solution η. Remark that we will be only interested in situations, in which $\lim_{t\to\infty} \eta(t) = \infty$ with positive probability and such that the infinity will not be reached in a finite time.

The main problem in this section is to study the conditions, under which

$$\lim_{t\to\infty} \frac{\eta(t)}{\mu(t)} = 1 \ \text{ a.s. on the set } \left\{\lim_{t\to\infty} \eta(t) = \infty\right\}, \tag{6}$$

where $\mu = (\mu(t), t \geq 0)$ is the continuous solution of the Cauchy problem for the ODE corresponding to (5) with $\sigma \equiv 0$, i.e.

$$d\mu(t) = g\left(\mu(t)\right)\varphi(t)dt, \quad t \geq 0; \tag{7}$$
$$\mu(0) = b > 0.$$

We assume that the functions g and φ are such that the continuous solution μ exists and satisfies the relation

$$\lim_{t\to\infty} \mu(t) = \infty.$$

For $t \geq 0$, we denote

$$\Phi(t) = \int_0^t \varphi(u)du \quad \text{and} \quad \Phi_+(t) = \int_0^t |\varphi(u)|\, du.$$

We assume that

$$\Phi(t) > 0, \ t > 0; \quad \lim_{t\to\infty} \Phi(t) = \infty, \tag{8}$$

and

$$\limsup_{t\to\infty} \frac{\Phi_+(t)}{\Phi(t)} < \infty\,. \tag{9}$$

Note that conditions (8) and (9) hold for any positive-valued function φ. An example of the alternating function φ, which satisfies conditions (8) and (9), is the function

$$\varphi(t) = a + \sin t, \quad t \geq 0,\ a \in (0,1)\,.$$

Put

$$G(x) = \int_b^x \frac{ds}{g\,(s)}, \quad x \geq b.$$

Then, for the solution μ of Eq. (7), one has the relation

$$G(\mu(t)) = \Phi(t), \quad t \geq 0.$$

Thus,

$$\mu(t) = G^{-1}(\Phi(t)), \quad t \geq 0, \tag{10}$$

where G^{-1} is the inverse function of G.

Consider the condition

$$\lim_{x\to\infty} G(x) = \infty. \tag{11}$$

Theorem 2.1. *Let φ and θ be continuous functions, and let g and σ be continuous positive functions such that* (5) *has a continuous solution η and conditions* (8), (9), *and* (11) *hold. Assume that*

$$\sum_{n=0}^{\infty} \frac{\int_{2^n}^{2^{n+1}} \theta^2(s)ds}{\Phi_+^2(2^n)} < \infty, \tag{12}$$

and also two following conditions hold:

a) *the function σ/g is bounded*;

b) *the function g is continuously differentiable, and its derivative $g'(x), x \in \mathbf{R}$, is such that*

$$\lim_{t\to\infty} \frac{\int_0^t g'(\eta(s))\theta^2(s)ds}{\Phi_+(t)} = 0 \quad \text{a.s. on the set} \quad \left\{\lim_{t\to\infty} \eta(t) = \infty\right\}. \tag{13}$$

Then

$$\lim_{t\to\infty} \frac{G(\eta(t))}{\Phi(t)} = 1 \quad \text{a.s. on the set} \quad \left\{\lim_{t\to\infty} \eta(t) = \infty\right\}.$$

Remark 2.1. *It is appropriate to use, instead of* (13), *the conditions*

$$\lim_{x\to\infty} g'(x) = 0 \quad \text{and} \quad \limsup_{t\to\infty} \frac{\int_0^t \theta^2(s)ds}{\Phi_+(t)} < \infty$$

or

$$\limsup_{x\to\infty} |g'(\mathrm{x})| < \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{\int_0^t \theta^2(s)ds}{\Phi_+(t)} = 0.$$

In the next theorem, we consider the conditions, under which the solutions of SDE (5) and its corresponding ODE (7) are asymptotically equivalent a.s..

Theorem 2.2. *Assume that all conditions of Theorem* 2.1 *hold, and*

$$\liminf_{t\to\infty} \int_t^{ct} \frac{du}{g(u)G(u)} > 0 \quad \text{for all} \quad c > 1. \tag{14}$$

Then (6) *holds.*

Remark 2.2. *If $\varphi(t) > 0$, $t \geq 0$, then all corresponding results of paper* [2] *follow from Theorems* 2.1–2.2. *Moreover, all corresponding results of papers* [5]–[7] *follow from these theorems with $\varphi(t) = \theta(t) = 1$, $t \geq 0$.*

Recall the conditions, under which condition (14) holds (see [6], [8]).

Proposition 2.1. *Let g be a positive continuous function such that* (11) *holds. Assume that at least one of the following conditions holds*:
1) $\limsup_{x\to\infty}(g(x)G(x)/x) < \infty$;
2) *g is eventually nonincreasing*;
3) *there exists $\alpha < 1$ such that* $0 < \inf_{x\ge 1} g(x)x^{-\alpha}$, $\sup_{x\ge 1} g(x)x^{-\alpha} < \infty$;
4) *$g^*(c) < c$ for all $c > 1$, with* $g^*(c) = \limsup_{x\to\infty}(g(cx)/g(x))$;
5) *g is a regular varying function with index $\alpha < 1$* (see [11]).

Then condition (14) *is satisfied.*

Now we consider the following SDE which has a new term on the right-hand side in contrast to SDE (5):

$$d\zeta(t) = \left(\tilde{g}\left(\zeta(t)\right)\varphi(t) + \tilde{g}_1(\zeta(t))\theta^2(t)\right)dt + \tilde{\sigma}\left(\zeta(t)\right)\theta(t)dw(t), \quad t \ge 0; \qquad (15)$$
$$\zeta(0) \equiv b > 0,$$

where w is a standard Wiener process; b is a nonrandom positive constant; $\tilde{g}_1$, θ, and φ are continuous functions, $\tilde{g}$ and $\tilde{\sigma}$ are continuous positive functions such that SDE (15) has a continuous solution ζ.

Two following auxiliary lemmas are needed to prove Theorem 2.1.

Lemma 2.1. *Assume that conditions* (8), (9), and (12) *and the following three conditions hold*:

A1) $\lim_{x\to\infty} \tilde{g}(x) = \varkappa \in (0, \infty)$;

B1) *the function $\tilde{\sigma}$ is bounded*;

C1)
$$\lim_{t\to\infty} \frac{\int_0^t \tilde{g}_1(\zeta(s))\theta^2(s)ds}{\Phi_+(t)} = 0 \quad \text{a.s. on the set} \quad \left\{\lim_{t\to\infty}\zeta(t) = \infty\right\}.$$

Then
$$\lim_{t\to\infty} \frac{\zeta(t)}{\Phi(t)} = \varkappa \quad \text{a.s. on the set} \quad \left\{\lim_{t\to\infty}\zeta(t) = \infty\right\}.$$

Lemma 2.2. *Let φ and θ be continuous functions, and let g and σ be continuous positive functions such that* (5) *has a continuous solution η and conditions* (8), (9), *and* (12) *hold. Assume that there is an increasing twice continuously differentiable function $f = (f(x),\, x \in \mathbf{R})$, for which three following conditions hold*:

A2) $\lim_{x\to\infty} f(x) = \infty$ *and* $\lim_{x\to\infty} f'(x)g(x) = C \in (0, \infty)$;

B2) *the function $f'\sigma$ is bounded*;

C2)
$$\lim_{t\to\infty} \frac{\int_0^t f''(\eta(s))\sigma^2\left(\eta(s)\right)\theta^2(s)ds}{\Phi_+(t)} = 0 \quad \text{a.s. on the set} \quad \left\{\lim_{t\to\infty}\eta(t) = \infty\right\}.$$

Then
$$\lim_{t\to\infty} \frac{f(\eta(t))}{\Phi(t)} = C \quad \text{a.s. on the set} \quad \left\{\lim_{t\to\infty}\eta(t) = \infty\right\}.$$

3. Asymptotic Equivalence of Solutions of ODE's

In this section, we consider ODE's (2) and discuss the condition, under which the solutions μ_1 and μ_2 of these ODE's are asymptotically equivalent, i.e. (3) holds true.

We assume that, for each $k = 1, 2$, the function g_k in (2) is continuous and positive on $(0, \infty)$, and the function φ_k is continuous.

For $k = 1, 2$, we put

$$G_k(x) = \int_{b_k}^x \frac{ds}{g_k(s)}, \ x \ge b_k; \ \Phi_k(t) = \int_0^t \varphi_k(u)du, \ (\Phi_k)_+(t) = \int_0^t |\varphi_k(u)|du, \ t \ge 0.$$

In the sequel, we make use four following conditions: for $k = 1, 2$,

$$\lim_{t\to\infty} G_k(t) = \infty; \tag{16}$$

$$\Phi_k(t) > 0,\ t \geq 0, \quad \text{and} \quad \lim_{t\to\infty} \Phi_k(t) = \infty; \tag{17}$$

$$\liminf_{t\to\infty} \int_t^{ct} \frac{du}{g_k(u) G_k(u)} > 0 \quad \text{for all} \quad c > 1; \tag{18}$$

$$\lim_{c\downarrow 1} \limsup_{t\to\infty} \int_t^{ct} \frac{du}{g_k(u) G_k(u)} = 0. \tag{19}$$

Theorem 3.1. *Let g_k and φ_k, $k = 1, 2$, be such that conditions* (16) and (17) *hold and also*

$$\lim_{t\to\infty} \frac{\Phi_1(t)}{\Phi_2(t)} = 1. \tag{20}$$

Then,

1) *if condition* (18) *holds for at least one of $k = 1, 2$, then*

$$\lim_{t\to\infty} \frac{G_1(t)}{G_2(t)} = 1 \quad \Longrightarrow \quad \lim_{t\to\infty} \frac{\mu_1(t)}{\mu_2(t)} = 1;$$

2) *if condition* (19) *holds for at least one of $k = 1, 2$, then*

$$\lim_{t\to\infty} \frac{G_1(t)}{G_2(t)} = 1 \quad \Longleftarrow \quad \lim_{t\to\infty} \frac{\mu_1(t)}{\mu_2(t)} = 1;$$

3) *if condition* (18) *holds for at least one of $k = 1, 2$, and condition* (19) *holds for at least one of $k = 1, 2$, then*

$$\lim_{t\to\infty} \frac{G_1(t)}{G_2(t)} = 1 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{\mu_1(t)}{\mu_2(t)} = 1.$$

Theorem 3.2. *Let g_k and φ_k, $k = 1, 2$, be such that conditions* (16) and (17) *hold and also*

$$\lim_{t\to\infty} \frac{G_1(t)}{G_2(t)} = 1. \tag{21}$$

Then,

1) *if condition* (18) *holds for at least one of $k = 1, 2$, then*

$$\lim_{t\to\infty} \frac{\Phi_1(t)}{\Phi_2(t)} = 1 \quad \Longrightarrow \quad \lim_{t\to\infty} \frac{\mu_1(t)}{\mu_2(t)} = 1;$$

2) *if condition* (19) *holds for at least one of $k = 1, 2$, then*

$$\lim_{t\to\infty} \frac{\Phi_1(t)}{\Phi_2(t)} = 1 \quad \Longleftarrow \quad \lim_{t\to\infty} \frac{\mu_1(t)}{\mu_2(t)} = 1;$$

3) *if condition* (18) *holds for at least one of $k = 1, 2$, and also condition* (19) *holds for at least one of $k = 1, 2$, then*

$$\lim_{t\to\infty} \frac{\Phi_1(t)}{\Phi_2(t)} = 1 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{\mu_1(t)}{\mu_2(t)} = 1.$$

Recall the conditions, under which condition (19) holds (see [6], [8]).

Proposition 3.1. *Let g be a positive continuous function such that* (11) *holds. Assume that at least one of the following conditions holds*:

1) $\liminf_{x\to\infty}(g(x)G(x)/x) > 0$;
2) g *is eventually nondecreasing*;
3) *there exists* $\alpha < 1$ *such that* $0 < \inf_{x\geq 1} g(x)x^{-\alpha}$, $\sup_{x\geq 1} g(x)x^{-\alpha} < \infty$;
4) $g^*(c) < c$ *for all* $c > 1$, *with* $g^*(c) = \limsup_{x\to\infty}(g(cx)/g(x))$;
5) g *is a regular varying function with index* $\alpha < 1$ (see [11]).

Then condition (19) *is satisfied with* $g_k = g$.

4. Asymptotic Equivalence of Solutions of SDE's

In this section, we consider SDE's (1) and discuss, under which conditions (4) holds true. This problem is the last one in the list of our problems (see Section 1), and its solution follows from the results of Sections 2 and 3, since

$$\frac{\eta_1(t)}{\eta_2(t)} = \frac{\eta_1(t)}{\mu_1(t)} \cdot \frac{\mu_1(t)}{\mu_2(t)} \cdot \frac{\mu_2(t)}{\eta_2(t)}. \tag{22}$$

We assume that φ_k and θ_k, $k = 1, 2$, are real continuous functions, and g_k and σ_k, $k = 1, 2$, are positive continuous functions such that, for each $k = 1, 2$, SDE (1) has a continuous Itô-solution η_k. Moreover, we assume that, for each $k = 1, 2$, the function g_k has the derivative $g'_k(x), x \in \mathbf{R}$.

In this section, we use four following conditions: for $k = 1, 2$,

$$\limsup_{t\to\infty} \frac{(\Phi_k)_+(t)}{\Phi_k(t)} < \infty; \tag{23}$$

$$\sum_{n=0}^{\infty} \frac{\int_{2^n}^{2^{n+1}} \theta_k^2(s)ds}{(\Phi_k)_+^2(2^n)} < \infty; \tag{24}$$

$$\text{the function} \quad \frac{\sigma_k}{g_k} \quad \text{is bounded;} \tag{25}$$

$$\lim_{t\to\infty} \frac{\int_0^t g'_k(\eta_k(s))\theta_k^2(s)ds}{(\Phi_k)_+(t)} = 0 \quad \text{a.s. on the set} \quad \left\{\lim_{t\to\infty} \eta_k(t) = \infty\right\}. \tag{26}$$

Theorem 4.1. *Assume that, for each $k = 1, 2$, conditions* (16), (17), (20), *and* (23)–(26) *hold. Then*

1) *if condition* (18) *holds for at least one of $k = 1, 2$, and if condition* (21) *holds, then* (4) *follows, i.e.*

$$\lim_{t\to\infty} \frac{\eta_1(t)}{\eta_2(t)} = 1 \quad \textit{a.s. on the set} \quad \bigcap_{k=1}^{2} \left\{\lim_{t\to\infty} \eta_k(t) = \infty\right\};$$

2) *if condition* (19) *holds for at least one of $k = 1, 2$, and if* (4) *holds with*

$$\mathbf{P}\left(\bigcap_{k=1}^{2} \left\{\lim_{t\to\infty} \eta_k(t) = \infty\right\}\right) > 0, \tag{27}$$

then (21) *follows*;

3) *if* (27) *holds and if condition* (18) *holds for at least one of $k = 1, 2$, and if condition* (19) *holds for at least one of $k = 1, 2$, then*

$$\lim_{t\to\infty} \frac{G_1(t)}{G_2(t)} = 1 \Longleftrightarrow \lim_{t\to\infty} \frac{\eta_1(t)}{\eta_2(t)} = 1 \quad \textit{a.s. on the set} \quad \bigcap_{k=1}^{2} \left\{\lim_{t\to\infty} \eta_k(t) = \infty\right\}.$$

Theorem 4.2. *Assume that, for each $k=1,2$, conditions* (16), (17), (21), *and* (23)–(26) *hold. Then*

1) *if condition* (18) *holds for at least one of $k=1,2$, and if condition* (20) *holds, then* (4) *follows, i.e.*

$$\lim_{t\to\infty}\frac{\eta_1(t)}{\eta_2(t)}=1 \quad \textit{a.s. on the set} \quad \bigcap_{k=1}^{2}\left\{\lim_{t\to\infty}\eta_k(t)=\infty\right\};$$

2) *if condition* (19) *holds for at least one of $k=1,2$, and if conditions* (4) *and* (27) *hold, then* (20) *follows*;

3) *if* (27) *holds and if condition* (18) *holds for at least one of $k=1,2$, and if condition* (19) *holds for at least one of $k=1,2$, then*

$$\lim_{t\to\infty}\frac{\Phi_1(t)}{\Phi_2(t)}=1 \Longleftrightarrow \lim_{t\to\infty}\frac{\eta_1(t)}{\eta_2(t)}=1 \quad \textit{a.s. on the set} \quad \bigcap_{k=1}^{2}\left\{\lim_{t\to\infty}\eta_k(t)=\infty\right\}.$$

5. Proofs of the Main Results

Proof of Lemma 2.1. We have

$$\zeta(t)=\zeta(0)+\int_0^t\tilde g\left(\zeta(s)\right)\varphi(s)ds+\int_0^t\tilde g_1\left(\zeta(s)\right)\theta^2(s)ds+\int_0^t\tilde\sigma\left(\zeta(s)\right)\theta(s)dw(s).$$

To prove Lemma 2.1, it is sufficient to show, by conditions (9) and C1), that

$$\lim_{t\to\infty}\frac{1}{\Phi(t)}\int_0^t\tilde g(\zeta(s))\varphi(s)ds=\varkappa \quad \text{a.s. on the set} \quad \left\{\lim_{t\to\infty}\zeta(t)=\infty\right\} \tag{28}$$

and

$$\lim_{t\to\infty}\frac{1}{\Phi_+(t)}\int_0^t\tilde\sigma(\zeta(s))\theta(s)dw(s)=0 \quad \text{a.s..} \tag{29}$$

By A1), we have that, for any $\omega\in\{\lim_{t\to\infty}\zeta(t)=\infty\}$ and any $\varepsilon>0$, there exists $s_\varepsilon=s_\varepsilon(\omega)>0$ such that $|\tilde g(\zeta(s))-\varkappa|\le\varepsilon$ for $s\ge s_\varepsilon$. Therefore, for any $t\ge s_\varepsilon$,

$$\frac{\left|\int_{s_\varepsilon}^t(\tilde g(\zeta(s))-\varkappa)\varphi(s)ds\right|}{\Phi(t)}\le\frac{\varepsilon\int_{s_\varepsilon}^t|\varphi(s)|\,ds}{\Phi(t)}\le\varepsilon\left(\frac{\Phi_+(t)}{\Phi(t)}\right)$$

and, in view of (8),

$$\limsup_{t\to\infty}\frac{\left|\int_0^t(\tilde g(\zeta(s))-\varkappa)\varphi(s)ds\right|}{\Phi(t)}=\limsup_{t\to\infty}\frac{\left|\int_{s_\varepsilon}^t(\tilde g(\zeta(s))-\varkappa)\varphi(s)ds\right|}{\Phi(t)}$$
$$\le\varepsilon\left(\limsup_{t\to\infty}\frac{\Phi_+(t)}{\Phi(t)}\right).$$

Hence, by (9), relation (28) holds.

In order to prove (29), we consider, for any $n\ge0$ and $\varepsilon>0$, the following two events:

$$B_n=\left\{\sup_{2^n\le t\le 2^{n+1}}\frac{1}{\Phi_+(t)}\left|\int_0^t\tilde\sigma(\zeta(s))\theta(s)dw(s)\right|>\varepsilon\right\}$$

and

$$C_n=\left\{\sup_{2^n\le t\le 2^{n+1}}\frac{1}{\Phi_+(2^n)}\left|\int_0^t\tilde\sigma(\zeta(s))\theta(s)dw(s)\right|>\varepsilon\right\}.$$

Since Φ_+ is an increasing function, $B_n\subset C_n$, $n\ge0$. Hence, by (Theorem 1, §3, [9]),

$$\mathbf{P}\left\{\sup_{2^n\le t\le 2^{n+1}}\frac{1}{\Phi_+(t)}\left|\int_0^t\tilde\sigma(\zeta(s))\theta(s)dw(s)\right|>\varepsilon\right\}$$

$$\leq \mathbf{P}\left\{\sup_{2^n\leq t\leq 2^{n+1}} \frac{1}{\Phi_+(2^n)}\left|\int_0^t \tilde{\sigma}(\zeta(s))\theta(s)dw(s)\right| > \varepsilon\right\}$$

$$\leq \frac{4}{\Phi_+^2(2^n)\varepsilon^2}\int_{2^n}^{2^{n+1}} E\left|\tilde{\sigma}(\zeta(s))\right|^2\theta^2(s)ds \leq \frac{4M^2\left(\int_{2^n}^{2^{n+1}}\theta^2(s)ds\right)}{\Phi_+^2(2^n)\varepsilon^2},$$

where $M = \sup_x \tilde{\sigma}(x) < \infty$.

Thus,

$$\mathbf{P}\left\{\sup_{2^n\leq t\leq 2^{n+1}} \frac{1}{\Phi_+(t)}\left|\int_0^t \tilde{\sigma}(\zeta(s))\theta(s)dw(s)\right| > \varepsilon\right\} \leq \frac{4M^2\left(\int_{2^n}^{2^{n+1}}\theta^2(s)ds\right)}{\Phi_+^2(2^n)\varepsilon^2} \tag{30}$$

for any $n \geq 0$ and $\varepsilon > 0$.

Now, for any $m \geq 1$ and $\varepsilon > 0$, we consider the following event:

$$\tilde{B}_m = \left\{\sup_{t\geq 2^m} \frac{1}{\Phi_+(t)}\left|\int_0^t \tilde{\sigma}(\zeta(s))\theta(s)dw(s)\right| > \varepsilon\right\}.$$

It is clear that $\tilde{B}_m = \bigcup_{k=m}^{\infty} B_k$. Therefore, by (30),

$$\mathbf{P}\left\{\sup_{t\geq 2^m} \frac{1}{\Phi_+(t)}\left|\int_0^t \tilde{\sigma}(\zeta(s))\theta(s)dw(s)\right| > \varepsilon\right\}$$

$$\leq \sum_{n=m}^{\infty} \mathbf{P}\left\{\sup_{2^n\leq t\leq 2^{n+1}} \frac{1}{\Phi_+(t)}\left|\int_0^t \tilde{\sigma}(\zeta(s))\theta(s)dw(s)\right| > \varepsilon\right\} \leq \Xi_m,$$

where

$$\Xi_m = \sum_{n=m}^{\infty} \frac{\int_{2^n}^{2^{n+1}}\theta^2(s)ds}{\Phi_+^2(2^n)}, \quad m \geq 1.$$

Thus, for any $\varepsilon > 0$ and $m \geq 1$,

$$\mathbf{P}\left\{\sup_{t\geq 2^m} \frac{1}{\Phi_+(t)}\left|\int_0^t \tilde{\sigma}(\zeta(s))\theta(s)dw(s)\right| > \varepsilon\right\} \leq \frac{4\Xi_m M^2}{\varepsilon^2}. \tag{31}$$

By condition A1), $\Xi_m \to 0$ as $m \to \infty$. Hence, by (31),

$$P\left\{\lim_{m\to\infty}\sup_{t\geq 2^m} \frac{1}{\Phi_+(t)}\left|\int_0^t \tilde{\sigma}(\zeta(s))\theta(s)dw(s)\right| > \varepsilon\right\}$$

$$= \lim_{m\to\infty} \mathbf{P}\left\{\sup_{t\geq 2^m} \frac{1}{\Phi_+(t)}\left|\int_0^t \tilde{\sigma}(\zeta(s))\theta(s)dw(s)\right| > \varepsilon\right\}$$

$$\leq \lim_{m\to\infty} \frac{4\Xi_m M^2}{\varepsilon^2} = 0$$

for any $\varepsilon > 0$. Therefore,

$$\lim_{t\to\infty} \frac{1}{\Phi_+(t)}\left|\int_0^t \tilde{\sigma}(\zeta(s))\theta(s)dw(s)\right| = 0 \quad \text{a.s.},$$

i.e. relation (29) holds, and Lemma 2.1 is proved. □

Proof of Lemma 2.2. Denote $\zeta(t) = f(\eta(t))$, $t > 0$. Then $\eta(t) = f^{-1}(\zeta(t))$, $t > 0$, where f^{-1} is the inverse function for f. It is clear that

$$\left\{\lim_{t\to\infty} \eta(t) = \infty\right\} = \left\{\lim_{t\to\infty} \zeta(t) = \infty\right\}.$$

Using the Itô formula for $f(\eta(t))$, we obtain

$$d\zeta(t) = [f'(\eta(t))g\left(\eta(t)\right)\varphi(t) + \frac{1}{2}f''(\eta(t))\sigma^2(\eta(t))\theta^2(t)]dt + f'(\eta(t))\sigma\left(\eta(t)\right)\theta(t)dw(t)$$

$$= f'(f^{-1}(\zeta(t))g\left(f^{-1}(\zeta(t))\right)\varphi(t) + \frac{1}{2}f''(f^{-1}(\zeta(t)))\sigma^2(f^{-1}(\zeta(t))\theta^2(t)]dt$$
$$+f'(f^{-1}(\zeta(t))\sigma\left(f^{-1}(\zeta(t))\right)\theta(t)dw(t).$$

Thus, the process ζ is a solution of the SDE

$$d\zeta(t) = (\tilde{g}(\zeta(t))\varphi(t) + \tilde{g}_1(\zeta(t))\theta^2(t))dt + \tilde{\sigma}(\zeta(t))\theta(t)dw(t),$$

where

$$\tilde{g}(x) = f'(f^{-1}(x))g(f^{-1}(x)),$$
$$\tilde{g}_1(x) = \frac{1}{2}f''(f^{-1}(x))\sigma^2(f^{-1}(x)),$$
$$\tilde{\sigma}(x) = f'(f^{-1}(x))\sigma\left(f^{-1}(x)\right).$$

Note that this SDE is similar to SDE (15). Hence, we can use Lemma 2.1.

It follows from conditions A2) and C2) that

$$\lim_{x\to\infty}\tilde{g}(x) = \lim_{x\to\infty} f'(f^{-1}(x))g(f^{-1}(x)) = \lim_{x\to\infty} f'(x)g(x) = C$$

and

$$\lim_{t\to\infty}\frac{\int_0^t \tilde{g}_1\left(\zeta(s)\right)\theta^2(s)ds}{\Phi_+(t)} = \frac{1}{2}\lim_{t\to\infty}\frac{\int_0^t f''(\eta(s))\sigma^2(\eta(s))\theta^2(s)ds}{\Phi_+(t)} = 0$$

a.s. on the set $\{\lim_{t\to\infty}\eta(t) = \infty\}$. Moreover, it follows from B2) that $\tilde{\sigma}$ is a bounded function. Thus, all conditions of Lemma 2.1 hold, and, therefore,

$$\lim_{t\to\infty}\frac{f(\eta(t))}{\Phi(t)} = C \quad \text{a.s. on the set} \quad \left\{\lim_{t\to\infty}\eta(t) = \infty\right\}.$$

□

Proof of Theorem 2.1. Let

$$f(x) = \int_0^x \frac{du}{g(u)},\ x \ge 0, \quad \text{and} \quad f(x) = -\int_x^0 \frac{du}{g(u)},\ x < 0.$$

Hence, $f' = 1/g$, $\lim_{x\to\infty} f'(x)g\,(x) = 1$, and the function $f'\sigma$ is bounded by condition a).

Further, it follows from condition b) that f is a twice continuously differentiable function, and, by a) and b),

$$\lim_{t\to\infty}\frac{\left|\int_0^t f''(\eta(s))\sigma^2(\eta(s))\theta^2(s)ds\right|}{\Phi_+(t)} = \lim_{t\to\infty}\frac{\left|\int_0^t \frac{\sigma^2(\eta(s))}{g^2(\eta(s))}\cdot g'(\eta(s))\theta^2(s)ds\right|}{\Phi_+(t)}$$
$$\le \lim_{t\to\infty}\frac{L\left|\int_0^t g'(\eta(s))\theta^2(s)ds\right|}{\Phi_+(t)} = 0$$

a.s. on the set $\{\lim_{t\to\infty}\eta(t) = \infty\}$, where

$$L = \sup_x \frac{\sigma^2(x)}{g^2(x)} < \infty.$$

Hence, all conditions of Lemma 2.2 hold with $C = 1$ and, therefore,

$$\lim_{t\to\infty}\frac{G(\eta(t))}{\Phi(t)} = \lim_{t\to\infty}\frac{f(\eta(t))}{\Phi(t)} = 1 \quad \text{a.s. on the set} \quad \left\{\lim_{t\to\infty}\eta(t) = \infty\right\}.$$

Thus, Theorem 2.1 is proved. □

Proof of Theorem 2.2. By Theorem 2.1,

$$\lim_{t\to\infty}\frac{G(\eta(t))}{\Phi(t)} = 1 \quad \text{a.s. on the set} \quad \left\{\lim_{t\to\infty}\eta(t) = \infty\right\}.$$

Moreover, by conditions (11) and (14) and Lemma 4.3 [6], with $f = G$ and $f' = 1/g$, we have

$$\liminf_{t\to\infty} \frac{G(ct)}{G(t)} > 1 \quad \text{for all} \quad c > 1.$$

Hence, by Theorem 3.2 [6] (see also [3]–[5]), the function G^{-1} preserves the asymptotic equivalence of the functions. Therefore,

$$\lim_{t\to\infty} \frac{\eta(t)}{\mu(t)} = \lim_{t\to\infty} \frac{G^{-1}(G(\eta(t)))}{G^{-1}(\Phi(t))} = 1$$

a.s. on the set $\{\lim_{t\to\infty} \eta(t) = \infty\}$. Thus, relation (6) holds, and Theorem 2.2 is proved. □

Proof of Theorem 3.1. Assume that (18) holds for at least one of $k = 1, 2$. Let, for example, $k = 1$. Then, by conditions (16) and (18) and Lemma 4.3 [6], with $f = G_1$ and $f' = 1/g_1$, we have

$$\liminf_{t\to\infty} \frac{G_1(ct)}{G_1(t)} > 1 \quad \text{for all} \quad c > 1.$$

Hence, by Theorem 3.2 [6] (see also [3]–[5]), the function G_1^{-1} preserves the asymptotic equivalence of the functions. Therefore, in view of (17) and (20),

$$\lim_{t\to\infty} \frac{G_1^{-1}(\Phi_1(t))}{G_1^{-1}(\Phi_2(t))} = 1.$$

If

$$\lim_{t\to\infty} \frac{G_1(t)}{G_2(t)} = 1,$$

then, by Theorem 3.3 [6] (see also [3]–[5]), we have

$$\frac{G_1^{-1}(t)}{G_2^{-1}(t)} = 1.$$

Therefore, in view of (17),

$$\lim_{t\to\infty} \frac{G_1^{-1}(\Phi_2(t))}{G_2^{-1}(\Phi_2(t))} = 1.$$

Finally, we obtain

$$\lim_{t\to\infty} \frac{\mu_1(t)}{\mu_2(t)} = \lim_{t\to\infty} \frac{G_1^{-1}(\Phi_1(t))}{G_2^{-1}(\Phi_2(t))} = \lim_{t\to\infty} \left(\frac{G_1^{-1}(\Phi_1(t))}{G_1^{-1}(\Phi_2(t))} \cdot \frac{G_1^{-1}(\Phi_2(t))}{G_2^{-1}(\Phi_2(t))} \right)$$

$$\lim_{t\to\infty} \frac{G_1^{-1}(\Phi_1(t))}{G_1^{-1}(\Phi_2(t))} \cdot \lim_{t\to\infty} \frac{G_1^{-1}(\Phi_2(t))}{G_2^{-1}(\Phi_2(t))} = 1,$$

since $\mu_j = G_j^{-1}(\Phi_j)$, $j = 1, 2$. Thus, statement 1) is proved.

In order to prove statement 2), we assume that (19) holds for at least one of $k = 1, 2$. Let, for example, $k = 1$. We assume also that relation (3) holds, i.e.

$$\lim_{t\to\infty} \frac{\mu_1(t)}{\mu_2(t)} = 1.$$

Note that $\lim_{t\to\infty} \mu_j(t) = \infty$, $j = 1, 2$, since (16) and (17) hold.

By condition (16) and (19) and Lemma 4.1 [6], with $f = G_1$ and $f' = 1/g_1$, we have

$$\lim_{c\to 1} \limsup_{t\to\infty} \frac{G_1(ct)}{G_1(t)} = 1 \quad \text{for all} \quad c > 1.$$

Hence, by Lemma 3.1 [6] (see also [3]–[5]), the function G_1 preserves the asymptotic equivalence of the functions. Moreover, in view of (3), we have

$$\lim_{t\to\infty} \frac{G_1(\mu_1(t))}{G_1(\mu_2(t))} = 1.$$

This relation in combination with (20) yield

$$\lim_{t\to\infty} \frac{G_1(\mu_2(t))}{G_2(\mu_2(t))} = 1,$$

since $\Phi_j = G_j(\mu_j)$, $j = 1, 2$. Therefore,

$$\lim_{t\to\infty} \frac{G_1(t)}{G_2(t)} = 1,$$

since the continuous function μ_2 goes to infinity, as $t \to \infty$. Thus, statement 2) is proved.

Statement 3) follows from statements 1) and 2). □

Proof of Theorem 3.2. The proof of Theorem 3.2 is similar to the proof of Theorem 3.1. □

Proof of Theorem 4.1. Theorem 4.1 follows from Theorems 2.2 and 3.1. □

Proof of Theorem 4.2. Theorem 4.1 follows from Theorems 2.2 and 3.2. □

References

1. V. V. Budygin and O. A. Tymoshenko, *On the asymptotic stability of stochastic differential equations*, Naukovi Visti NTUU "KPI" (2007), no 1, 126-129.
2. V. V. Budygin and O. A. Tymoshenko, *The exact order of grouth of solutions of stochastic differential equations*, Naukovi Visti NTUU "KPI" (2008), no 6, 127-132.
3. V. V. Buldygin, O. I. Klesov, and J. G. Stainebach, *Properties of a subclass of Avakumović functions and their generalized inverses*, Ukrain. Mat. Zh. **54** (2002), 179–206.
4. V. V. Buldygin, O. I. Klesov, and J. G. Stainebach, *On some properties of asymptotically quasi-inverse functions and their applications. I*, Theory Probab. Math. Statist. **70** (2005), 11–28.
5. V. V. Buldygin, O. I. Klesov, and J. G. Stainebach, *On some properties of asymptotically quasi-inverse functions and their applications. II*, Theory Probab. Math. Statist. **71**(2005), 37–52.
6. V. V. Buldygin, O. I. Klesov, and J. G. Stainebach, *The PRV property of functions and the asymptotic behaviour of solutions of stochastic differential equations*, Theory Probab. Math. Statist. **72**(2006), 11-25.
7. V. V. Buldygin, O. I. Klesov, and J. G. Steinebach, *PRV property and the $\boldsymbol{\varphi}$–asymptotic behaviour of solutions of stochastic differential equations*, Lithuanian Math. J. (2007), no 4, 1–21.
8. V. V. Budygin, O. I. Klesov, J.G. Steinebach, and O. A. Tymoshenko, *On the $\boldsymbol{\varphi}$-asymptotic behaviour of solutions of stochastic differential equations*, Theory Stoch. Process. **14**(2008), 11-29.
9. I. I. Gikhman, A. V. Skorokhod, *Stochastic Differential Equations*, Springer, Berlin, 1972.
10. G. Keller, G. Kersting, and U. Rösler, *On the asymptotic behaviour of solutions of stochastic differential equations*, Z. Wahrsch. verw. Geb. **68**(1984), 163–184.
11. E. Seneta, *Regularly Varying Functions*, Springer, Berlin, 1976.

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