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ON ASYMPTOTIC BEHAVIOR OF THE ERROR TERM IN CROSS-CORRELOGRAM ESTIMATION OF RESPONSE FUNCTIONS IN LINEAR SYSTEMS

The problem of estimation of an unknown response function of a linear system with inner noises is considered. We suppose that the response function of the system belongs to $L_2(\mathbb{R})$. Integral-type sample input-output cross-correlograms are taken as estimators of the response function. The inputs are supposed to be zero-mean stationary Gaussian processes that are close, in some sense, to a white noise. Both the asymptotic normality of finite-dimensional distributions of the normalized error term in the cross-correlogram estimation and the asymptotic normality in the space of continuous functions are discussed.

1. Introduction

We consider a time-invariant casual continuous linear Volterra system with inner noises and a response function $H = (H(\tau), \tau \in \mathbb{R})$. This means that the real-valued function H satisfies the condition $H(\tau) = 0,\ \tau < 0$, and the response of the system to an input process $X(t),\ t \in \mathbb{R}$, has the form

$$U(t) = \int_0^\infty H(\tau)X(t-\tau)\,d\tau + Z(t),\ t \in \mathbb{R}, \tag{1}$$

where the process $Z(t),\ t \in \mathbb{R}$, describes inner noises of the system.

Let us focus on the problem of estimation of the unknown function H by observations of responses of the system to certain input signals. To solve this problem, a lot of deterministic methods exist, as well as statistical approaches. The latter are based on a perturbation of the system by stationary stochastic processes and the further analysis of some characteristics of both input and output processes [3, 5, 13]. For the estimation of the stability or instability of the system, the methods of periodograms or cross-correlograms may be useful (see [1, 4] or [6, 9], respectively). In the cross-correlogram method, the sample correlograms between input and output processes are taken as estimators for H. Such an approach is suitable, when the input process is close, in some sense, to the Gaussian white noise ([8], [10]–[12]).

In work [7], we used the method of integral-type correlograms for the estimation of the response function $H \in L_2(\mathbb{R})$. Both the asymptotic normality of finite-dimensional distributions of the centered estimators and their asymptotic normality in the space of continuous functions were studied.

This paper continues the research started in [7] and contains the final results about the asymptotic normality of the normalized error term in the cross-correlogram estimation of H.

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2. Preliminaries

Assume that $X_\Delta = (X_\Delta(t),\ t \in \mathbb{R})$, $\Delta > 0$, is a family of measurable real-valued stationary zero-mean Gaussian processes that disturb system (1). Let $f_\Delta = (f_\Delta(\lambda),\ \lambda \in \mathbb{R})$, $\Delta > 0$, be a family of spectral densities of the processes X_Δ. We suppose that these functions are nonnegative and continuous and satisfy the conditions

$$f_\Delta(\lambda) = f_\Delta(-\lambda),\ \lambda \in \mathbb{R}; \tag{2a}$$

$$\sup_{\Delta>0} \|f_\Delta\|_\infty < \infty; \tag{2b}$$

$$f_\Delta \in L_1(\mathbb{R}); \tag{2c}$$

$$\exists c \in (0,\infty)\ \ \forall a \in (0,\infty):\ \lim_{\Delta\to\infty} \sup_{-a\le\lambda\le a} \left| f_\Delta(\lambda) - \frac{c}{2\pi} \right| = 0; \tag{2d}$$

$$K_\Delta \in L_1(\mathbb{R}), \tag{2e}$$

where $K_\Delta(t) = \int_{-\infty}^{\infty} e^{i\lambda t} f_\Delta(t)\, dt,\ t \in \mathbb{R}$, is the correlation function of X_Δ.

By (1), the reaction of the system to an input signal X_Δ is represented by

$$U_\Delta(t) = \int_0^\infty H(\tau) X_\Delta(t-\tau)\, d\tau + Z(t),\ t \in \mathbb{R}. \tag{3}$$

We assume that the inner noise $(Z(t),\ t \in \mathbb{R})$ is a separable real-valued stationary zero-mean Gaussian process which is orthogonal to X_Δ; that is, $\mathbf{E}X_\Delta(s)Z(t) = 0,\ s,t \in \mathbb{R}$.

Let $(g(\lambda),\ \lambda \in \mathbb{R})$ be the spectral density of the process Z. It is a nonnegative measurable function which satisfies the conditions

$$g(\lambda) = g(-\lambda); \tag{4a}$$

$$g \in L_1(\mathbb{R}). \tag{4b}$$

The so-called *cross-correlogram* (or *the sample cross-correlation function*)

$$\widehat{H}_{T,\Delta}(\tau) = \frac{1}{cT} \int_0^T U_\Delta(t+\tau) X_\Delta(t)\, dt,\ \tau \ge 0, \tag{5}$$

will be used as an estimator for H. Here, c is the constant from (2d), and T is the length of the averaging interval. The integrals in (3) and (5) are interpreted as a mean square Riemann integrals.

Generally speaking, for all $T > 0, \Delta > 0$, and $\tau \ge 0$,

$$H(\tau) \ne \mathbf{E}\widehat{H}_{T,\Delta}(\tau) = \frac{1}{c} \int_{-\infty}^{\infty} K_\Delta(\tau - s) H(s)\, ds,$$

that is, the estimator $\widehat{H}_{T,\Delta}$ is biased.

Consider the normalized error term

$$\widehat{W}_{T,\Delta}(\tau) = \sqrt{T}[\widehat{H}_{T,\Delta}(\tau) - H(\tau)],\ \tau \ge 0. \tag{6}$$

The further results deal with asymptotic properties of $\widehat{W}_{T,\Delta} = (\widehat{W}_{T,\Delta}(\tau), \tau \ge 0)$ as the parameters T, Δ tend to infinity. Let us represent (6) as the sum

$$\widehat{W}_{T,\Delta}(\tau) = A_{T,\Delta}(\tau) + B_{T,\Delta}(\tau),\ \tau \ge 0, \tag{7}$$

where

$$A_{T,\Delta}(\tau) = \sqrt{T}\Big[\widehat{H}_{T,\Delta}(\tau) - \mathbf{E}\widehat{H}_{T,\Delta}(\tau)\Big]; \tag{8}$$

$$B_{T,\Delta}(\tau) = \sqrt{T}\Big[\mathbf{E}\widehat{H}_{T,\Delta}(\tau) - H(\tau)\Big]. \tag{9}$$

From (7), the asymptotic properties of the process $\widehat{W}_{T,\Delta}$ are characterized by properties of the stochastic process $A_{T,\Delta} = (A_{T,\Delta}(\tau), \tau \geq 0)$ and the function $B_{T,\Delta} = (B_{T,\Delta}(\tau), \tau \geq 0)$ as T, Δ tend to infinity.

Now we eliminate the dependence of $B_{T,\Delta}$ in the representation of $\widehat{W}_{T,\Delta}$. For this purpose, note some conditions on the order of local smoothness of H and the character of tending T, Δ to infinity (see [8]).

Let $\alpha \in (0,1]$. We say that $H \in Lip_\alpha[0,\infty)$, if there exist constants $\delta > 0$ and $M > 0$ such that

$$\forall t, s \geq 0 \ \ \exists \delta > 0: \ |t-s| < \delta \Rightarrow |H(t) - H(s)| < M|t-s|^\alpha.$$

(That is, H is uniformly on $[0,\infty)$ satisfies the Lipschitz condition with index α.)

Example 2.1. (A) Let $H(\tau) = \frac{\cos \mu\tau}{(1+\tau)^\beta}$, $\tau \geq 0$, where $\mu > 0$ and $\beta \in \left(0, \frac{1}{2}\right)$. Then $H \in Lip_1[0,\infty) \cap L_2(\mathbb{R})$ (here, the Lipschitz constant $M = \mu + \beta$), and its Fourier–Plancherel transform has the form

$$H^*(\lambda) = \begin{cases} \frac{1}{2}\left[e^{-i(\lambda+\mu)} \int_1^\infty \frac{e^{i(\lambda+\mu)\tau}}{\tau^\beta}\, d\tau + e^{-i(\lambda-\mu)} \int_1^\infty \frac{e^{i(\lambda-\mu)\tau}}{\tau^\beta}\, d\tau\right], & \lambda \neq \pm\mu, \\ +\infty, & \lambda = \pm\mu. \end{cases}$$

(B) Let $H(\tau) = \frac{1}{1+\tau}$, $\tau \geq 0$. Then $H \in Lip_\alpha[0,\infty) \cap L_2(\mathbb{R})$ for $\alpha \in (0,1]$ (here, Lipschitz constant $M = \left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{1}{1-\alpha}}$ for $\alpha \in (0,1)$ or $M = 1$ for $\alpha = 1$, respectively), and its Fourier–Plancherel transform has the form

$$H^*(\lambda) = \begin{cases} e^{-i\lambda} \int_1^\infty \frac{e^{i\lambda\tau}}{\tau}\, d\tau, & \lambda \neq 0, \\ +\infty, & \lambda = 0; \end{cases}$$

(C) Let $H(\tau) = \frac{1+\tau}{1+\tau^2}$, $\tau \geq 0$. Then $H \in Lip_\alpha[0,\infty) \cap L_2(\mathbb{R})$ for $\alpha \in (0,1]$ (here, the Lipschitz constant $M = \|\frac{1-2x-x^2}{(1+x^2)^2}\|_{\frac{1}{1-\alpha}}$ for $\alpha \in (0,1)$ or $M = 2\frac{1}{4}$ for $\alpha = 1$, respectively), and its Fourier-Plancherel transform has the form

$$H^*(\lambda) = \begin{cases} \int_0^\infty e^{i\lambda\tau} \frac{1+\tau}{1+\tau^2}\, d\tau, & \lambda \neq 0, \\ +\infty, & \lambda = 0. \end{cases}$$

Given $\alpha \in (0,1]$. Assume that $T \to \infty, \Delta \to \infty$ in such a way that

$$\sqrt{T}\left[1 - \frac{2\pi f_\Delta(0)}{c}\right] \to 0; \tag{10a}$$

$$\forall \delta > 0: \sqrt{T} \int_\delta^\infty K_\Delta(t)\, dt \to 0; \tag{10b}$$

$$\forall \delta > 0: T \int_\delta^\infty K_\Delta^2(t)\, dt \to 0; \tag{10c}$$

$$\exists \delta > 0: \sqrt{T} \int_{-\delta}^\delta |K_\Delta(t)||t|^\alpha\, dt \to 0. \tag{10d}$$

Example 2.2. Let $\alpha \in (0,1]$ and $H \in Lip_\alpha[0,\infty) \cap L_2(\mathbb{R})$. The spectral densities f_Δ and the correlation functions K_Δ of the processes X_Δ are

(A) $f_\Delta = \left(\frac{c}{2\pi} \exp\left(-\frac{\lambda^2}{\Delta}\right), \lambda \in \mathbb{R}\right)$ and $K_\Delta = \left(\frac{c}{2}\sqrt{\frac{\Delta}{\pi}} \exp\left(-\frac{\Delta t^2}{4}\right),\ t \in \mathbb{R}\right)$;

(B) $f_\Delta = \left(\frac{c}{2\pi} \frac{\Delta}{\Delta+\lambda^2}, \lambda \in \mathbb{R}\right)$ and $K_\Delta = \left(c\sqrt{\Delta} \exp\left(-\sqrt{\Delta}t\right),\ t \in \mathbb{R}\right)$, and satisfy conditions (2a) - (2e) and (10a) - (10d), if $T \to \infty, \Delta \to \infty$ in such a way that

$$T\Delta^{-\alpha} \to 0.$$

Further, we will use the following assertion (see [8]):

Lemma 2.1. *Let* $\alpha \in (0,1]$*;* $H \in Lip_\alpha[0,\infty) \cap L_2(\mathbb{R})$ *and* $T \to \infty, \Delta \to \infty$ *in such a way that conditions (10a) - (10d) hold true. Then*
(i) $\forall \tau \geq 0$ $\quad B_{T,\Delta}(\tau) \to 0$*;*
(ii) $\forall a > 0$ $\quad \sup_{\tau \in [0,a]} |B_{T,\Delta}(\tau)| \to 0.$

In work [7], it was shown that if $H \in L_2(\mathbb{R})$ and $g \in L_1(\mathbb{R})$, then the correlation function of $A_{T,\Delta}$ for any $\tau_1, \tau_2 \geq 0$ has the form

$$\mathbf{E}A_{T,\Delta}(\tau_1)A_{T,\Delta}(\tau_2) = \frac{2\pi}{c^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[e^{i(\tau_1-\tau_2)\lambda_2}\left(|H^*(\lambda_2)|^2 f_\Delta(\lambda_2) + g(\lambda_2)\right) + \tag{11}$$

$$+e^{i(\tau_1\lambda_1+\tau_2\lambda_2)}H^*(\lambda_1)H^*(\lambda_2)f_\Delta(\lambda_2)\Big]\Phi_T(\lambda_2-\lambda_1)f_\Delta(\lambda_1)\,d\lambda_1 d\lambda_2,$$

where Φ_T is the Fejer kernel; that is,

$$\Phi_T(\lambda) = \frac{1}{2\pi T}\left(\frac{\sin(T\lambda/2)}{\lambda/2}\right)^2, \ \lambda \in \mathbb{R},$$

and H^* is the Fourier-Plancherel transform of H in $L_2(\mathbb{R})$.

The limit $C_\infty(\tau_1,\tau_2)$ of correlation function from (11) as $T, \Delta \to \infty$ has the following form:

$$C_\infty(\tau_1,\tau_2) = \lim_{T,\Delta\to\infty}\mathbf{E}A_{T,\Delta}(\tau_1)A_{T,\Delta}(\tau_2) = \tag{12}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[e^{i(\tau_1-\tau_2)\lambda}\left(|H^*(\lambda)|^2 + \frac{2\pi}{c}g(\lambda)\right) + e^{i(\tau_1+\tau_2)\lambda}(H^*(\lambda))^2\right]d\lambda.$$

3. Asymptotic behavior of the correlation function of $\widehat{W}_{T,\Delta}$

In this section, we consider the asymptotic behavior of the correlation function of $\widehat{W}_{T,\Delta}$ as T and Δ tend to infinity.

Theorem 3.1. *Assume that* $g \in L_1(\mathbb{R})$*; for some* $\alpha \in (0,1]$, *the response function* $H \in Lip_\alpha[0,\infty) \cap L_2(\mathbb{R})$ *and* $T \to \infty, \Delta \to \infty$ *in such a way that conditions (10a) - (10d) are satisfied. Then the relation*

$$\boldsymbol{E}\widehat{W}_{T,\Delta}(\tau_1)\widehat{W}_{T,\Delta}(\tau_2) \to C_\infty(\tau_1,\tau_2)$$

holds for all $\tau_1, \tau_2 \geq 0$.

Proof. The statement of Theorem 3.1 follows immediately from the representation

$$\mathbf{E}\widehat{W}_{T,\Delta}(\tau_1)\widehat{W}_{T,\Delta}(\tau_2) = B_{T,\Delta}(\tau_1)B_{T,\Delta}(\tau_2) + \mathbf{E}A_{T,\Delta}(\tau_1)A_{T,\Delta}(\tau_2),$$

Lemma 2.1 (part (i)), and formula (12). □

4. Asymptotic normality of finite-dimensional distributions of $\widehat{W}_{T,\Delta}$

Theorem 3.1 demonstrates that the function C_∞ defined in (11) is positive semi-definite on $[0,\infty)\times[0,\infty)$. So, there exists a zero-mean real-valued Gaussian process $A=(A(\tau),\tau\geq 0)$ with the correlation function C_∞; that is,

$$\mathbf{E}A(\tau_1)A(\tau_2)=C_\infty(\tau_1,\tau_2).$$

Without loss of generality, we assume that the process A is defined on the same probability space $\{\Omega,\mathfrak{F},\mathbb{P}\}$ as the processes $A_{T,\Delta}$ and $\widehat{W}_{T,\Delta}$.

Theorem 4.1. *Assume that $g\in L_1(\mathbb{R})$; for some $\alpha\in(0,1]$, the response function $H\in Lip_\alpha[0,\infty)\cap L_2(\mathbb{R})$ and $T\to\infty,\Delta\to\infty$ in such a way that conditions (10a) - (10d) are satisfied. Then the relation*

$$\boldsymbol{E}\left[\prod_{j=1}^{m}\widehat{W}_{T,\Delta}(\tau_j)\right]\to\boldsymbol{E}\left[\prod_{j=1}^{m}A(\tau_j)\right] \tag{13}$$

holds for any $m\in\mathbb{N}$ and any τ_1, ..., $\tau_m\geq 0$.

In particular, all finite-dimensional distributions of the process $(\widehat{W}_{T,\Delta}(\tau),\tau\geq 0)$ converge weakly to the corresponding finite-dimensional distributions of the Gaussian process $(A(\tau),\tau\geq 0)$ by the given character of tending T and Δ to infinity.

Remark 4.1. Theorem 4.1 refines results of [8] (see Theorem 3). To show that the analogous statements hold true, some additional assumptions on the Fourier–Plancherel transformation of the response function H are required, namely: 1) $H^*\in L_1(\mathbb{R})\cap L_\infty(\mathbb{R})$; 2) H^* is continuous almost everywhere on $\mathbb{R}$.

Proof. From representation (7), it follows that

$$\mathbf{E}\left[\prod_{j=1}^{m}\widehat{W}_{T,\Delta}(\tau_j)\right]=\mathbf{E}\sum_{k_1=0,k_2=0,\ldots,k_m=0}^{1}\left[\prod_{j=1}^{m}A_{T,\Delta}^{k_j}(\tau_j)B_{T,\Delta}^{1-k_j}(\tau_j)\right]=$$

$$=\sum_{k_1=0,k_2=0,\ldots,k_m=0}^{1}\left[\mathbf{E}\prod_{j=1}^{m}A_{T,\Delta}^{k_j}(\tau_j)\right]\prod_{j=1}^{m}B_{T,\Delta}^{1-k_j}(\tau_j).$$

By the given character of tending T and Δ to infinity, the last formula together with Lemma 2.1 (part (i)) and Theorem 4.1 [7] yield

$$\mathbf{E}\left[\prod_{j=1}^{m}\widehat{W}_{T,\Delta}(\tau_j)\right]\to\mathbf{E}\left[\prod_{j=1}^{m}A_{T,\Delta}(\tau_j)\right]\to\mathbf{E}\left[\prod_{j=1}^{m}A(\tau_j)\right]$$

for any $m\in\mathbb{N}$ and $\tau_j\geq 0$, $j=1,...,m$. So, we proved formula (13).

By the Markov theorem (see, [2]), the weak convergence of finite-dimensional distributions of the process $(\widehat{W}_{T,\Delta}(\tau),\tau\geq 0)$, to the corresponding finite-dimensional distributions of the process $(A(\tau),\tau\geq 0)$, takes place, since the Gaussian process $(A(\tau),\tau\geq 0)$ is uniquely determined by its moments. □

5. Asymptotic normality of $\widehat{W}_{T,\Delta}$ in the space of continuous functions

In addition to Theorem 4.1, it is natural to study the asymptotic normality of $\widehat{W}_{T,\Delta}$ in the space of continuous functions. Assume that $A_{T,\Delta},\widehat{W}_{T,\Delta}$, $T>0$, $\Delta>0$, and A are separable processes. We use the notation $C[0,a]$, $a>0$, for the space of real-valued continuous functions defined on $[0,a]$ and endowed with uniform norm. In what follows, we write $\widehat{W}_{T,\Delta} \overset{C[0,a]}{\Longrightarrow} A$ to denote the weak convergence of the process $\widehat{W}_{T,\Delta}$ to the process A in the space $C[0,a]$ by the given character of tending T and Δ to infinity.

We now recall some tools related to Gaussian stochastic processes (see, e.g., [9]). Let S be a parameter set. A function $\rho(t,s)$, $t,s \in S$, is called a pseudometric on S if it satisfies all axioms of a metric, with the exception that the set $\{(t,s) \in S \times S : \ \rho(t,s) = 0\}$ may be wider than the diagonal $\{(t,s) \in S \times S : \ t = s\}$. We write $N_\rho(S,\varepsilon)$ for the minimal number of closed ρ-balls of radius $\varepsilon > 0$, whose centers lie in S and which cover S. If there is no finite covering of S, then $N_\rho(S,\varepsilon) = \infty$. Further, let $\mathcal{H}_\rho(S,\varepsilon) = \log N_\rho(S,\varepsilon)$ be a metric entropy of the set S with respect to ρ. For any $\beta > 0$, the inequality $\int_{0+} \mathcal{H}_\rho^\beta(S,\varepsilon)\, d\varepsilon < \infty$ is always interpreted in the sense that, for some (and, hence, for all) $u > 0$, we have $\int_0^u \mathcal{H}_\rho^\beta(S,\varepsilon)\, d\varepsilon < \infty$.

Consider the function [7]

$$\sigma_{H,g}(\tau) = \left[\int_{-\infty}^{\infty} \sin^2 \frac{\tau\lambda}{2} (|H^*(\lambda)|^2 + g(\lambda))\, d\lambda\right]^{\frac{1}{2}}, \ \tau \geq 0.$$

Since $H \in L_2(\mathbb{R})$ and $g \in L_1(\mathbb{R})$, this function is well-defined and generates the following two pseudometrics: $\sigma(\tau_1,\tau_2) = \sigma_{H,g}(|\tau_1 - \tau_2|)$ and $\sqrt{\sigma}(\tau_1,\tau_2) = \sqrt{\sigma(\tau_1,\tau_2)}$, $\tau_1,\tau_2 \geq 0$. Note that if $H^*(\lambda) \neq 0$ and $g(\lambda) \neq 0$ simultaneously on the set of a positive Lebesgue measure, then σ and $\sqrt{\sigma}$ are metrics.

For all $\varepsilon > 0$, put $\mathcal{H}_\sigma(\varepsilon) = \mathcal{H}_\sigma([0,1],\varepsilon)$, $\mathcal{H}_{\sqrt{\sigma}}(\varepsilon) = \mathcal{H}_{\sqrt{\sigma}}([0,1],\varepsilon)$. Since the pseudometrics σ and $\sqrt{\sigma}$ depend on $|\tau_1 - \tau_2|$ only, one has, for any $a > 0$ and $\beta > 0$,

$$\int_{0+} \mathcal{H}_\sigma^\beta(\varepsilon)\, d\varepsilon < \infty \Longleftrightarrow \int_{0+} \mathcal{H}_\sigma^\beta([0,a],\varepsilon)\, d\varepsilon < \infty;$$

$$\int_{0+} \mathcal{H}_{\sqrt{\sigma}}(\varepsilon)\, d\varepsilon < \infty \Longleftrightarrow \int_{0+} \mathcal{H}_{\sqrt{\sigma}}([0,a],\varepsilon)\, d\varepsilon < \infty.$$

Theorem 5.1. *Assume that* $g \in L_1(\mathbb{R})$*; for some* $\alpha \in (0,1]$*, the response function* $H \in Lip_\alpha[0,\infty) \bigcap L_2(\mathbb{R})$*, and the condition*

$$\int_{0+} \mathcal{H}_{\sqrt{\sigma}}(\varepsilon)\, d\varepsilon < \infty, \tag{14}$$

is satisfied. Then, for any $a > 0$*, the following statements hold true:*

(I) $A \in C[0,a]$ *almost surely;*

(II) $\widehat{W}_{T,\Delta} \in C[0,a]$ *almost surely,* $T > 0, \Delta > 0$*;*

Moreover, if $T \to \infty$*,* $\Delta \to \infty$ *in such a way that conditions (10a) -(10d) are satisfied, then*

(III) $\widehat{W}_{T,\Delta} \overset{C[0,a]}{\Longrightarrow} A$*.*

In particular, by the given character of tending T *and* Δ *to infinity, for all* $x > 0$ *and* $a > 0$*,*

$$\mathbb{P}\left\{\sup_{\tau\in[0,a]} \left|\widehat{W}_{T,\Delta}(\tau)\right| > x\right\} \to \mathbb{P}\left\{\sup_{\tau\in[0,a]} |A(\tau)| > x\right\}.$$

Remark 5.1. Statement (I) of Theorem 5.1 holds true under a weaker condition than (14), namely,

$$\int_{0+} \mathcal{H}_\sigma^{\frac{1}{2}}(\varepsilon)\, d\varepsilon < \infty. \tag{15}$$

Note that (15) always holds if there exists $\beta > 0$ such that (see [14])

$$\int_0^\infty \left(|H^*(\lambda)|^2 + g(\lambda)\right) \log^{1+\beta}(1+\lambda)\, d\lambda < \infty.$$

Remark 5.2. Condition (14) holds if there exists $\beta > 0$ such that (see [12])

$$\int_0^\infty \left(|H^*(\lambda)|^2 + g(\lambda)\right) \log^{4+\beta}(1+\lambda)\, d\lambda < \infty.$$

Proof. Using formula (14), statement (I) was proved in paper [7] (see Theorem 5.1, part I)). The other statements of Theorem 5.1 immediately follow from formula (7), Lemma 2.1 (part (ii)), Theorem 5.1 [7] (parts II) and III)), and Theorem 4.1. □

Conclusion

This paper continues the research from [7] concerning the problem of the cross-correlogram estimation of an unknown response function of a linear system with inner noises. Main results are presented in Theorem 4.1 and Theorem 5.1 and deal with the asymptotic normality of finite-dimensional distributions of the estimates and their asymptotic normality in the space of continuous functions.

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