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INCLUSIONS FOR GENERATORS OF EVOLUTION FAMILIES AND STOCHASTIC FLOWS

Some inclusions for generators of Feller evolution families are considered. The existence of solutions of these inclusions is proved with the use of stochastic flows having the same generators as the evolution families.

We study the solvability of inclusions given in terms of generators of Feller evolution families. We consider the cases where it turns out that the right-hand side of such an inclusion has a selection generating unique Markov process. A more general case will be considered in future works.

1. Generators

1.1. **Generator of the evolution family of operators.** By definition, a function f on $\mathbb{R}^n$ belongs to $C_0(\mathbb{R}^n)$ if it is continuous and if, for every $\varepsilon > 0$, there exists a compact subset K_ε of $\mathbb{R}^n$ such that $\|f(x)\|_{\mathbb{R}^n} < \varepsilon$ for each $x \notin K_\varepsilon$.

We recall that the family of operators $U(s,t)$ (take $t \geq s$) on $C_0(\mathbb{R}^n)$ it is called the Feller evolution family if the following properties hold:

(1) the evolution property $U(s,\tau)U(\tau,t) = U(s,t)$ $(s \leq \tau \leq t)$ and $U(s,s) = I$;
(2) operators $U(s,t)$ acts in $C_0(\mathbb{R}^n)$: $U(s,t)(C_0(\mathbb{R}^n)) \subset C_0(\mathbb{R}^n)$;
(3) operators $U(s,t)$ are strongly continuous jointly in both parameters;
(4) for any $f \in C_0(\mathbb{R}^n)$, $0 \leq f \leq 1$, and $t \geq s \geq 0$, the inequality $0 \leq U(s,t)f \leq 1$ holds.

The infinitesimal generator of such a family is the operator $G(s,x)$ such that its action on every function from $C_0(\mathbb{R}^n)$ is given by the formula

$$G(s,x)f(x) = \lim_{t \downarrow s} \frac{U(t,s)f(x) - f(x)}{t-s}.$$

For more details, see, e.g., [1, 2].

1.2. **Generator of the stochastic flow.** Take mappings $a(t,x)$ and $A(t,x)$ from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$ and the set of linear operators on $\mathbb{R}^n$, respectively. Consider the stochastic dynamical system in $\mathbb{R}^n$ governed by an equation of the Itô type

$$\begin{cases} d\xi_{s,x}(t) = a(t,\xi_{s,x}(t))dt + A(t,\xi_{s,x}(t))dw(t), \\ \xi_{s,x}(s) = x. \end{cases} \tag{1}$$

given on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$, where $w(t)$ is a Wiener process on that space with values in $\mathbb{R}^n$; $0 \leq s \leq t \leq T$. We recall that the infinitesimal generator of this stochastic evolution family of processes is the operator $G(s,x)$ which acts on the functions $f \in C(\mathbb{R}^n, \mathbb{R})$ in the following way:

$$G(s,x)f(x) = \lim_{t \downarrow s} \frac{\mathsf{E}[f(\xi_{s,x}(t))] - f(x)}{t-s},$$

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where E is the expectation. Then, for $f \in C^2(\mathbb{R}^n, \mathbb{R})$, it can be written in the form

$$G(s,x) = a^i(s,x)\frac{\partial}{\partial q^i} + \frac{1}{2}(AA^*)^{ij}(s,x)\frac{\partial^2}{\partial q^i \partial q^j},$$

where q^i, $i = 1, 2, \dots, n$ are coordinates in $\mathbb{R}^n$.

By $\mu_{s,x}$, we denote the measures on the space of sample paths corresponding to the solutions $\xi_{s,x}(t)$ of (1).

2. Inclusion for the generator of the evolution family

We start this section by recalling some definitions from the set-valued analysis (see, e.g., [3]).

Let X and Y be metric spaces, and let F be a set-valued map from X to Y.

Definition 2.1. A set-valued map F is called lower semicontinuous at $x \in X$ if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that, for every x', from a δ-neighborhood of x, the image $F(x)$ belongs to an ε-neighborhood of $F(x')$.

Definition 2.2. Let X and Y be normed spaces. A set-valued map F from X to Y is called Lipschitz continuous at $x \in X$ if there exist $k > 0$ and neighborhood U of x such that

$$\forall x_1, x_2 \in U, \quad F(x_1) \subset F(x_2) + k\|x_1 - x_2\| B_Y,$$

where B_Y is a unit ball in Y. It is called Lipschitz continuous if it is Lipschitz continuous at each point $x \in X$, and the constant k is independent of x.

Denote the norm of a set-valued map in a standard way:

$$\|F(x)\| = \sup_{y \in F(x)} \|y\|.$$

Consider a set-valued semielliptic differential operator $\mathbf{G}(s,x)$ on $\mathbb{R} \times \mathbb{R}^n$. It can be written in the form

$$\mathbf{G}(s,x) = \mathbf{a}^i(s,x)\frac{\partial}{\partial q^i} + \boldsymbol{\alpha}^{ij}(s,x)\frac{\partial^2}{\partial q^i \partial q^j},$$

where $\mathbf{a}(s,x)$ and $\boldsymbol{\alpha}(s,x)$ are set-valued mappings from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$ and to $\bar{S}_+(n)$ (set of all positive semidefinite symmetric square matrices of dimension n), respectively.

Remark 2.1. To avoid misunderstandings, we will call a set or single-valued differential operator semielliptic if its second order part takes values in the space $\bar{S}_+(n)$ (i.e., matrices may degenerate) and elliptic if they lie in $S_+(n)$ (positive definite symmetric square matrices).

In $\mathbb{R}^n$, it is convenient to introduce the norm of the semielliptic differential operator $\boldsymbol{G}(s,x)$ as a sum of the norms of its components, i.e., the norm of a vector of the first order part plus the norm of a matrix of the second order part.

By I, we denote a closed interval in $\mathbb{R}$.

Theorem 2.1. *Suppose that the measurable set-valued semielliptic differential operator* $\boldsymbol{G}(s,x)$ *has a second-order term* $\alpha(s,x)$ *single-valued* C^2*-smooth in* x *and satisfies the inequality*

$$\|\boldsymbol{a}(s,x)\|^2 + \|\operatorname{tr}\alpha(s,x)\| \leqslant K(s)(1 + \|x\|^2) \tag{2}$$

for some function $K(s)$ *on* I*. Then there exists the Feller evolution family with generator* $G(s,x)$ *such that the inclusion*

$$G(s,x) \in \boldsymbol{G}(s,x) \tag{3}$$

holds for all (s,x) *from* $I \times \mathbb{R}^n$*.*

Proof We recall that there exists a measurable selection $G(s,x)$ of the set-valued map $\boldsymbol{G}(s,x)$ (see, e.g., [3]). Since $\boldsymbol{G}(s,x)$ is a semielliptic differential operator, its selection $G(s,x)$ can be written in the form

$$G(s,x) = a^i(s,x)\frac{\partial}{\partial q^i} + \alpha^{ij}(s,x)\frac{\partial^2}{\partial q^i \partial q^j}, \tag{4}$$

where $a(s,x)$ is measurable, and $\alpha(s,x)$ is the above-mentioned positive semidefinite symmetric C^2-smooth matrix. By [4, Theorem 1], there exists the Lipschitz continuous $A(s,x)$ such that $\alpha(s,x) = A(s,x)A^*(s,x)$. Since estimate (2) holds, the stochastic equation without drift

$$\mathrm{d}\tilde{\xi}_{s,x}(t) = A(t, \tilde{\xi}_{s,x}(t)\mathrm{d}w(t),$$

where $w(t)$ is a Wiener process in $\mathbb{R}^n$, has the unique strong solution for each initial data $\tilde{\xi}_{s,x}(s) = x$. Then, by results of [5], the equation with drift $a(t,x)$

$$\mathrm{d}\xi_{s,x}(t) = a(t, \xi_{s,x}(t)\mathrm{d}t + A(t, \xi_{s,x}(t)\mathrm{d}w(t),$$

has the unique weak solution for each initial data. Hence, the unique week solution $\xi(t)$, $t \in I$, of the equation

$$\mathrm{d}\xi(t) = a(t, \xi(t))\mathrm{d}t + A(t, \xi(t))\mathrm{d}w(t)$$

exists. It is such that, for $t \in I$, $t \geq s$, it coincides with $\xi_{s,x}(t)$ with probability 1.

This solution (see, e.g., [2]) is a Markov process. Thus, we get a dynamical system of form (1), and the operators $U(t,s)$ defined as $U(t,s)f(x) = \mathsf{E}(f(\xi_{s,x}(t)))$ form a Feller evolution family with the generator $G(s,x)$. So, inclusion (3) holds for each $(s,x) \in I \times \mathbb{R}^n$. □

Theorem 2.2. *Suppose that a set-valued elliptic differential operator $\boldsymbol{G}(s,x)$ is such that $\mathbf{a}(s,x)$ and $\boldsymbol{\alpha}(s,x)$ are Lipschitz continuous (in set-valued sense), and their values belong to the sets of nonempty closed convex subsets of $\mathbb{R}^n$ and $S_+(n)$, respectively. Let also inequality* (2) *hold. Then there exists a Feller evolution family such that inclusion* (3) *holds for all (s,x) from $I \times \mathbb{R}^n$.*

Proof By [3, Theorem 9.4.3], there exists the Lipschitz continuous selection $G(s,x)$ of the set-valued map $\boldsymbol{G}(s,x)$. Obviously, it satisfies estimate (2). From the fact that $\boldsymbol{\alpha}(s,x)$ is non-degenerate, it follows that there exists the Lipschitz continuous $A(s,x)$ such that $A(s,x)A^*(s,x) = \sigma(s,x)$.

By the existence theorem on strong solutions (see, e.g., [5]), the equation

$$\mathrm{d}\xi_{s,x}(t) = a(t, \xi_{s,x}(t)\mathrm{d}t + A(t, \xi_{s,x}(t)\mathrm{d}w(t),$$

has the unique strong solution $\xi_{s,x}(t)$ starting at the moment s from the point x.

As in Theorem 2.1, we construct the Feller evolution family by the rule $U(t,s)f(x) = \mathsf{E}(f(\xi_{s,x}(t)))$. Its generator satisfies inclusion (3). □

Suppose that a set-valued elliptic differential operator $\boldsymbol{G}(s,x)$ is lower semicontinuous and has convex closed images. Then, by Michael's selection theorem (see, e.g., [3]), it has a continuous selection $G(s,x)$. Since $G(s,x)$ is elliptic, i.e., the matrix $\alpha(s,x)$ of the second-order part is non-degenerate, there exists the continuous matrix $A(s,x)$ such that $\alpha(s,x) = A(s,x)A^*(s,x)$ (see [6]). This matrix $A(s,x)$ together with the first-order term $a(s,x)$ of $G(s,x)$ determine the stochastic differential equation of form (1) with continuous coefficients. Let inequality (2) hold. Then the above equation has a weak solution for each initial data for $t \in I$, $t \geq s$. Assume also that those solutions are weakly unique.

Theorem 2.3. *Under the above assumptions, there exists a Feller evolution family such that inclusion* (3) *holds for all (s,x) from $I \times \mathbb{R}^n$.*

Proof. Since the above-mentioned solutions are weakly unique, they are Markov processes. So, the family $U(t,s)f(x) = \mathsf{E}(f(\xi_{s,x}(t)))$ is an evolution one (see [2]), whose generator satisfies (3) by construction. □

Some parabolic differential inclusions. As in, e.g., [7], consider a function $u(s,x)$, $s \in \mathbb{R}$, $x \in \mathbb{R}^n$. Let $\boldsymbol{L}(s,x)$ be a set-valued second-order semielliptic differential operator. Consider a parabolic differential inclusion of the following type

$$\frac{\partial}{\partial s} u(s,x) \in \boldsymbol{L}(s,x)u(s,x). \tag{5}$$

For any initial value $u(t,x) = \phi(x)$, the solution $u(s,x)$ of (5) can be constructed as $u(s,x) = U(t,s)\phi(x)$, where $U(t,s)$ is the Feller evolution family.

So it is possible to find a solution $u(t,x)$ of (5) if it is possible to find a solution of inclusion (3).

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