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A BRIEF OVERVIEW OF THE L_p-THEORY OF SPDES

We present basics of the L_p-theory of SPDEs and its connection to various related problems of filtration and populational dynamics.

1. Motivation

For the author the main motivation to study SPDEs comes from filtering theory for diffusion processes. Imagine that we are given a two component diffusion process

$$z_t = (x_t, y_t) \in \mathbb{R}^d \times \mathbb{R}^{d_1}$$

such that we only can observe y_t and need to extract from the observations as much information about x_t as possible. Then it turns out that under natural assumptions for measurable bounded $f(x)$ we have

$$E(f(x_t)|y_s, s \le t) = \int_{\mathbb{R}^d} f(x) p_t(x)\, dx,$$

where the posterior density $p_t(x)$ of x_t given $y_s, s \le t$, satisfies an equation like

$$dp_t = L_t p_t\, dt + M_t p_t\, dy_t, \tag{1.1}$$

where L_t is a second-order elliptic operator, M_t is a first-order operator, and the coefficients of L_t and M_t depend on (t, x) and y_t.

Equation (1.1) is a natural generalization of the backward Kolmogorov equation for diffusion processes when $y_t \equiv 0$ and the second term on the right in (1.1) disappears.

Another natural source of SPDEs is populational dynamics or branching diffusion processes. These are measure-valued processes called super-diffusions and for a long time they were studied by using martingale approach and *nonstandard* analysis. If $d = 1$ since 1988 it is known that the super-diffusion μ_t has a density p_t which satisfies the equation

$$dp_t(x) = D^2 p_t(x)\, dt + p_t^{1/2}(x)\phi^k(x)\, dw_t^k, \tag{1.2}$$

where $D = \partial/\partial x$, ϕ^k form an orthonormal basis in L_2, w_t^k, $k = 1, 2, \dots$, are independent Wiener processes, and the summation convention is enforced. See [3], [17].

By the way, quite often in the literature after work by J. Walsh [18] in 1986 the second term on the right in (1.2) is written by using the so-called space-time white noise. The form (1.2) was probably first introduced by T. Funaki [2] in 1983 and turns out more convenient in many respects.

I will only concentrate on (1.1) and (1.2) without speaking about very active research areas related to random perturbation of deterministic equations like Navier-Stokes or wave equations or other types of equations. Part of really immense corresponding literature can be found following the references in the articles listed in the bibliography.

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2. Equations related to (1.2)

First I will discuss my articles related to super-diffusions: [8], [11].

Before 1997 there were two common ways known in the literature to prove the existence of super-diffusions. One of them is to take the limit law of branching diffusion particles which is analogous to proving the existence of diffusion processes not through solving stochastic Itô equations but through the passage to the weak limit in a sequence of Markov chains. Another way is to construct the super-diffusions as measure-valued Markov processes by defining their transition functions in the space of measures and then using the general theory of Markov processes to get a process corresponding to this transition function. This way is similar to the one used in the theory of diffusion processes with "bad" but not too "bad" coefficients when there are "good" results concerning the fundamental solutions for corresponding parabolic equations. The natural question arises: are there any stochastic Itô type equations for the super-diffusions at least for "regular" ones? In [8] I showed that the answer to this question is positive and the appropriate stochastic equations are stochastic partial differential equations. As is mentioned above this result was known for quite a while in one-dimensional case (see [3], [17]), and my contribution relates to the multidimensional case.

Answering our question we also answer the question concerning the possibility to include super-diffusions in the framework of more or less classical stochastic analysis, without resorting to the abstract theory of Markov processes or relying on nonstandard analysis.

The multidimensional equation should look like (1.2):

$$dp_t(x) = D^2 p_t(x)\,dt + p_t^{1/2}(x)\phi^k(x)\,dw_t^k.$$

However, in the general case the super-diffusions are measure-valued processes without densities and what is $\sqrt{\mu_t}$ for a measure-valued function μ_t is not immediately clear.

To give a better idea about the contents of [8] we introduce the following notation. By $\mathcal{M}$ we denote the space of all finite measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. One endows $\mathcal{M}$ with the usual measurable structure requiring functions (ψ, μ) to be measurable for any $\psi \in C_0^\infty(\mathbb{R}^d)$, where

$$(\psi, \mu) = \int_{\mathbb{R}^d} \psi(x)\,\mu(dx).$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with an increasing filtration of complete σ–algebras $\mathcal{F}_t \subset \mathcal{F}$, $t \geq 0$. An $\mathcal{M}$–valued $\mathcal{F}_t$–adapted process μ_t is called a super–Brownian process if for any $\psi \in C_0^\infty(\mathbb{R}^d)$ the process (ψ, μ_t) is continuous and the process

$$m_t(\psi) := (\psi, \mu_t) - (\psi, \mu_0) - \int_0^t (\Delta\psi, \mu_s)\,ds \tag{2.1}$$

is a continuous local martingale starting from zero with

$$d\langle m(\psi)\rangle_t = (\psi^2, \mu_t)\,dt.$$

Then to derive a stochastic equation for μ_t we need to find a "canonical" representation for the local martingales $m_t(\psi)$. If we could show that for some nonrandom real-valued functions $m^k(\mu) = m^k(\mu, x)$ we have

$$(\psi^2, \mu) = \sum_k (\psi, m^k(\mu))^2 = \Big(\int_{\mathbb{R}^d} \psi(x) m^k(\mu, x)\,dx\Big)^2, \tag{2.2}$$

then it is clear that

$$dm_t(\psi) = (\psi, m^k(\mu_t))\,dw_t^k,$$

where w_t^k are independent Wiener processes. Coming back to (2.1) we would therefore find

$$d\mu_t = \Delta\mu_t\,dt + m^k(\mu_t)\,dw_t^k. \tag{2.3}$$

Thus, the whole point was to find representation (2.2). If μ has a density p, then this is easy. Indeed,

$$(\psi^2,\mu) = \int_{\mathbb{R}^d}(p^{1/2}\psi)^2\,dx = \sum_k(\psi,\phi^k p^{1/2})^2,$$

where ϕ^k is any orthonormal basis in L_2. In this way one obtains (1.2). In the general case finding formulas like (2.2) is more difficult. In that article there are also derived equations for super-diffusions with jumps.

Observe that (2.3) contains a series of stochastic integrals and not a stochastic integral against a space-time white noise. If $d=1$ one still can use space-time white noises, but, most likely, it is impossible in higher dimensions.

In [8] I derived equations for super-diffusions assuming that they exist. In [11] a way to prove the existence is presented on the basis of showing that my equations have solutions. Here I used the Skorokhod embedding method.

In [9] we consider the following one-dimensional stochastic partial differential equation

$$\begin{aligned} dp(t,x) &= \nu(t,x)p^\lambda(t,x)\phi^k(x)\,dw_t^k \\ &+(a(t,x)D^2p(t,x)+b(t,x)Dp(t,x)+c(t,x)p(t,x))\,dt, \end{aligned} \tag{2.4}$$

where $\lambda\in(0,1)$, the coefficients a,b,c,ν are random and satisfy some natural conditions, the function $p=p(\omega,t,x)$ is supposed to be nonnegative. Actually, equation (2.4) has to be understood in a certain generalized sense since, for instance, the series of $\varphi^k(x)w_t^k$ diverges for almost all x.

In the case that $a=1, b=c=0$ C. Mueller and E. Perkins in [16] considered nonnegative solutions and by using *nonstandard* analysis proved that if the initial data has compact support, then this is true for all times. We prove the same for (2.4) by using the L_p-theory of SPDEs, which started with my article [7] the results of which without proofs are published as [6]and more complete exposition of which is in [10].

In each of these articles we generalize a result by Mueller [15] who proved that, for $\lambda\in[1,3/2)$, the following simplified equation (2.4):

$$du_t = D^2u_t\,dt + u_+^\lambda\phi^k\,dw_t^k \tag{2.5}$$

has a unique solution defined for all t if the initial condition u_0 is nonnegative and, say, is nonrandom and belongs to C_0^∞. Furthermore,

$$\sup_{t\le T,x}|u(t,x)|<\infty$$

(a.s.) for any $T<\infty$. We prove these facts for equation (2.4) with random and space and time dependent coefficients. In [15] quite a different method is used. The emphasis in [15] and [7] is on the fact that u_+^γ is super-linear in u.

3. An overview of the L_p-theory in the whole space

Here we are dealing with the following general equation

$$\begin{aligned} du_t &= [\sigma_t^{ik}D_iu_t+\nu_t^ku_t+g_t^k]\,dw_t^k \\ &+[(1/2)a_t^{ij}D_{ij}u_t+b_t^iD_iu_t+c_tu_t+f_t]\,dt, \end{aligned} \tag{3.1}$$

where $a^{ij}, f, \sigma^{ik}, g^k$ are real-valued functions defined for $\omega \in \Omega$, $t \geq 0$, $x \in \mathbb{R}^d$, $u \in H_p^{n+2}$, $i,j = 1,...,d$, $k = 1,2,...$, n is a fixed real number. The definition of spaces H_p^{n+2} is given later. The main assumption is that the matrix

$$(a^{ij} - \alpha^{ij}), \quad \text{where} \quad \alpha^{ij} = \sigma^{ik}\sigma^{jk},$$

is bounded and uniformly nondegenerate.

As it is mentioned above the development of the L_p-theory started with [7] (or [6]). Before this article various aspects of the L_2-theory for general equations were investigated by Pardoux, Krylov, Rozovskii, Gyongy, Flandoli, Brzezniak, and few other researchers.

The interest in L_p-theory is easy to explain, for instance, from computational point of view. If we want to know how fast, say finite-difference approximating schemes will converge to the true solution of an SPDE, we need to know how smooth the true solution is. In the framework of the L_2-theory in order to guarantee that the solution has one continuous derivative in x, we need u to have $[d/2]+2$ generalized derivatives summable to the second power. This requires the coefficients to have $[d/2]$ derivatives in x. In the framework of the L_p-theory the true solution is continuously differentiable in x if the coefficients of the equation are merely continuous (σ, ν Lipschitz continuous) and $p > d$. From the L_p-theory with $p > 2$ one also gets Hölder continuity of solutions in t. This result is either impossible or very hard to obtain in the framework of the L_2-theory.

Passing from the Hilbert space L_2 to L_p with $p \geq 2$ required proving one crucial estimate, which was done in [5].

Recall that the space $H_p^\gamma = H_p^\gamma(\mathbb{R}^d)$, $p \in (1,\infty)$, $\gamma \in (-\infty,\infty)$, of Bessel potentials is defined as the closure of $C_0^\infty = C_0^\infty(\mathbb{R}^d)$ with respect to the norm

$$\|u\|_{H_p^\gamma} = \|(I-\Delta)^{\gamma/2}u\|_{L_p}.$$

Consider the simplest one-dimensional SPDE

$$du_t(x) = \frac{1}{2}D^2u_t(x)\,dt + g_t(x)\,dw_t \tag{3.2}$$

given for $t > 0$, $x \in \mathbb{R}^d$, with initial condition $u_0 = 0$, where w_t is a one-dimensional Wiener process. The solution of this problem is known to be

$$u_t(x) = \int_0^t T_{t-s}g_s(x)\,dw_s, \tag{3.3}$$

where $T_th(x) = Eh(x+w_t)$ is the heat semi-group. If g is non random, then $u_t(x)$ is a Gaussian random variable with zero mean and its absolute moments are just powers of its second moment. It follows that

$$E\int_0^T ||u_t||_{H_p^1}^p\,dt = N(p)\int_0^T \int_{\mathbb{R}^d}[\int_0^t |(I-\Delta)^{1/2}T_{t-s}g_s(x)|^2\,ds]^{p/2}\,dx\,dt,$$

and in order to prove that $u \in H_p^1$ we have to estimate the last integral. The result is that

$$\int_0^\infty \int_{\mathbb{R}^d}[\int_0^t |DT_{t-s}g_s(x)|^2\,ds]^{p/2}\,dx\,dt \leq N\|g\|_{L_p((0,\infty)\times\mathbb{R}^d)}^p.$$

This result is also published in [13] where complete proofs of everything are given from scratch.

Now we describe the function spaces in which (3.2) is treated. For a stopping time τ we denote

$$\mathbb{H}_p^\gamma(\tau) = L_p(\rrbracket\, 0, 2\,\rrbracket), \mathcal{P}, H_p^\gamma), \quad \mathbb{H}_p^\gamma = \mathbb{H}_p^\gamma(\infty), \quad \mathbb{L}_{\dots}\dots = \mathbb{H}_{\dots}^0\dots.$$

For distributions u_t, f_t, g_t^k, depending on t and ω and measurable in a natural sense, we write

$$du_t = f_t\,dt + g_t^k\,dw_t^k, \;\; t > 0, \tag{3.4}$$

if for any $\phi \in C_0^\infty$ with probability 1 we have

$$(u_t, \phi) = (u_0, \phi) + \int_0^t (f_s, \phi)\, ds + \int_0^t (g_s^k, \phi)\, dw_s^k$$

for all $t \geq 0$ assuming that all integrals here make sense.

For a function $u \in \mathbb{H}_p^\gamma(\tau)$ we write $u \in \mathcal{H}_{p,0}^\gamma(\tau)$ if there exists $f \in \mathbb{H}_p^{\gamma-2}(\tau)$ $g \in \mathbb{H}_p^{\gamma-1}(\tau)$ such that equality (3.4) holds in the above sense of distributions. In this case we define

$$\|u\|_{\mathcal{H}_p^\gamma(\tau)} = \|u_{xx}\|_{\mathbb{H}_p^{\gamma-2}(\tau)} + \|f\|_{\mathbb{H}_p^{\gamma-2}(\tau)} + \|g\|_{\mathbb{H}_p^{\gamma-1}(\tau)}.$$

These spaces are convenient for treating equations with zero initial condition. The general case reduces to this one in a standard way.

Theorem 3.1. *Take a $p \geq 2$ and a bounded stopping time τ. Under the above condition of uniform nondegeneracy also suppose that the coefficients b, c, σ, ν are bounded and σ, ν satisfy the Lipschitz condition in x uniformly with respect to t, ω. Finally, assume that in (3.1):*

$$du_t = [(1/2)a_t^{ij} D_{ij} u_t + b_t^i D_i u_t + c_t u_t + f_t]\, dt + [\sigma_t^{ik} D_i u_t + \nu_t^k u_t + g_t^k]\, dw_t^k,$$

the summation in k is restricted to a finite set. Then for any $f \in \mathbb{L}_p(\tau)$ and $g \in \mathbb{H}_p^1(\tau)$ there exists a unique $u \in \mathcal{H}_{p,0}^2(\tau)$ satisfying (3.1). Furthermore,

$$\|u\|_{\mathcal{H}_p^2(\tau)} \leq N(\|f\|_{\mathbb{L}_p(\tau)} + \|g\|_{\mathbb{H}_p^1(\tau)}),$$

where N is independent of f and g.

There are a few extensions of this result when the right-hand sides f, g belong to spaces $\mathbb{H}$ and are nonlinear functions of u, for instance, like in (2.4), (2.5).

By the way, neither Theorem 3.1 nor its natural extension to nonzero initial-value problem are applicable to (2.5):

$$du_t = D^2 u_t\, dt + u_+^\lambda \phi^k\, dw_t^k,$$

since the range of the summation in k is $1, 2, \ldots$. Actually, the solutions of (2.5) never belong to $\mathcal{H}_p^2(\tau)$ but rather to $\mathcal{H}_p^\gamma(\tau)$, where γ is any number $< 1/2$. In particular, this shows the necessity to treat the equations in the spaces $\mathcal{H}_p^\gamma(\tau)$ with arbitrary $\gamma \in \mathbb{R}$.

4. SPDEs in half-spaces

It turns out that the theory of SPDEs in domains is much harder than in the whole space. For quite some time the only available results were obtained in function spaces with low regularity or under some compatibility conditions on the data. We refer to works by Pardoux, Flandoli, Brzezniak, Da Prato, Zabczyk, and others.

Let us come back to the simplest equation (3.2):

$$du_t(x) = \frac{1}{2} D^2 u_t(x)\, dt + g_t(x)\, dw_t$$

again with zero initial condition but considered only on the half-line $\mathbb{R}_+ = (0, \infty)$ and with zero lateral condition. The solution is given by the same formula (3.3) but with T_t defined as the heat semigroup in $\mathbb{R}_+$ with zero lateral condition. In particular, when $g \equiv 1$

$$u_t(x) = \int_0^t T_{t-s} 1(x)\, dw_s.$$

It is very easy to prove that $u_t(x)$ is infinitely differentiable with respect to x in $\mathbb{R}_+$. However, its second-order derivative cannot be bounded near the origin. To understand that write

$$u_t(x) = (1/2) \int_0^t D^2 u_s(x)\, ds + w_t$$

at $x=0$. Then the left-hand side vanishes owing to the boundary condition and we see that w_t is equal to a function which is differentiable in t, which is impossible.

This is the reason why we needed to introduce function spaces where elements are allowed to have derivatives blowing up near the boundary.

In 1994 in my article [4] such spaces were introduced and used to develop the Hilbert space theory for SPDEs in bounded smooth domains.

In notation we introduce below the results of [4] allow us to obtain solutions of class $\mathfrak{H}^\gamma_{p,\theta}(\tau)$, where γ is an integer indicating the number of derivatives the functions of this class possess, the power of summability $p=2$, and the parameter θ controlling the blow up near the boundary equals d.

The spaces $\mathfrak{H}^\gamma_{p,\theta,0}(\tau)$ are introduced for all $\gamma,\theta\in\mathbb{R}$ and any domain. However, it is easier to explain what they are when $\gamma=2$ and the domain the equation is considered in is

$$\mathbb{R}^d_+=\{x\in\mathbb{R}^d : x^1>0\}.$$

Observe that again restricting to the case that γ is an integer does not allow one to treat equations from populational dynamics like (1.2).

We say that $u\in\mathfrak{H}^2_{p,\theta,0}(\tau)$ if (3.4):

$$du_t=f_t\,dt+g^k_t\,dw^k_t,\ \ t>0,$$

holds in the sense of distributions on $\mathbb{R}^d_+$, $u_0=0$ and u, f, and g are such that

$$E\int_0^\tau\int_{\mathbb{R}^d_+}(x^1)^{\theta-d}(|(x^1)^{-1}u_t(x)|^p+|Du_t(x)|^p+|x^1D^2u_t(x)|^p\,dxdt<\infty,$$

$$E\int_0^\tau\int_{\mathbb{R}^d_+}(x^1)^{\theta-d}(|x^1f_t(x)|^p+|g_t(x)|^p_{\ell_2}+|x^1Dg_t(x)|^p_{\ell_2})\,dxdt<\infty.$$

It turns out that for the purpose of solving, say uniformly nondegenerate SPDEs the range of θ should be restricted to

$$d-1<\theta<p+d-1.$$

If $\theta\le d-1$, then generally solutions do not belong to $\mathfrak{H}^2_{p,\theta,0}(\tau)$, and if $\theta\ge d+p-1$, then generally one cannot find any solution at all. For θ so restricted the presence of $(x^1)^{\theta-d}|(x^1)^{-1}u_t|^p$ in the above conditions shows that $u_t(x)$ vanishes in an integral sense as $x^1\downarrow 0$. Therefore, while solving equations in $\mathfrak{H}^2_{p,\theta,0}(\tau)$, we are solving them with zero lateral condition.

For the same reasons as above we needed better results than in [4] in what concerns the power p of summability of derivatives.

One more drawback of the Hilbert space theory is that, in the general case, no matter to which class $\mathfrak{H}^\gamma_{2,\theta}(\tau)$ a function u belongs with as large γ as you wish one still cannot conclude that

$$u_t(x)\to 0\quad\text{as}\quad x\downarrow 0,\tag{4.1}$$

that is that the boundary condition is satisfied pointwise. One needs this kind of global continuity of solutions, for instance, for numerical approximations. Proving (4.1) became a major challenge for the theory. The current state of the art is that (4.1) is finally established in my paper [14] for the one-dimensional case assuming that the coefficients are *independent* of x. In addition the convergence in (4.1) is shown to become extremely slow as the constant of nondegeneracy of the equation becomes small.

This behavior is absolutely different from what is happening with ordinary parabolic equations. Typically their solutions decay linearly as x approaches the boundary regardless of the size of the constant of ellipticity.

The main nontrivial ingredient in [14] is a new square root law for one-dimensional Brownian motion discovered in my article [12].

It is worth saying that one of closely related square root laws was proved by B. Davis in 1983, see [1].

It says that (a.s.) if $c < 1$, then

(a) there exists a $t \in [0,1]$ and an $\varepsilon > 0$ such that

$$w_{t+r} - w_t \geq c\sqrt{r} \quad \text{for all} \quad r \in [0, \varepsilon]$$

but if $c > 1$, then the opposite is true, that is

(b) for any $t \in [0,1]$ and any $\varepsilon > 0$ there is an $r \in (0, \varepsilon]$ such that

$$w_{t+r} - w_t < c\sqrt{r}.$$

The new law can be described in the following way. Let $c \in (0, \infty)$ be a constant. For any $t \geq 0$ and $k = 0, 1, 2, ...$, we say that $w.$ after time t is contained in a (parabolic) c-box of size 2^{-k} if there is a number a such that

$$a \leq w_{t+s} \leq a + c2^{-k/2} \quad \text{for} \quad 0 \leq s \leq 2^{-k}. \tag{4.2}$$

The law of iterated logarithm applied to $w_{t+2^{-k}}$ implies that for each t (a.s.) there are infinitely many k's such that $w.$ after time t is *not* contained in c-boxes of size 2^{-k}. Actually, an even stronger statement is true. Observe that the sequence of processes $\xi_t(n) := 2^{n/2} w_{t2^{-n}}$, $t \in [0,1]$, $n = 0, 1, ...$, is a stationary $C([0,1])$-valued sequence. Therefore, the sequence

$$\Delta_n(0) = 2^{n/2} \text{osc}_{[0,2^{-n}]} w.$$

is stationary too and, by the law of large numbers, for each c, with probability one the density of k's, for which $w.$ after time 0 is not contained in c-boxes of size 2^{-k}, is strictly bigger than zero. (By the density of sets $A \subset \{0, 1, 2, ...\}$ we mean

$$\lim_{n \to \infty} \#(A \cap [0, n])/n,$$

where $\#B$ is the number of elements in B.)

Furthermore, Davis's law shows that if $c < 1$, then with probability one there is a $t = t(\omega) \in [0,1]$ such that $w.$ after time t is *not* contained in c-boxes of size 2^{-k} if k is large enough.

If c is large enough the situation is different. Our square root law says, in particular, that *there is a function $p(c) \geq 0$ such that $p(c) > 0$ if c is large enough, and (a.s.) for any $t \in [0,1]$, $w.$ after time t is contained in a c-box of size 2^{-k} for k's in a set with lower density $\geq p(c)$.*

It is worth noting again that if we are only interested in one particular value of t, then the result is a straightforward consequence of the above mentioned law of large numbers for stationary sequences. One need not even have c large enough, since (a.s.) the density of k's for which $w.$ after time t is contained in a c-box of size 2^{-k} equals the probability $p_0(c)$ that $w.$ after time 0 is contained in a c-box of size 1 and this probability is strictly bigger than zero for any $c > 0$. However, the exceptional set of ω depends on t and for $c < 1$ the union of exceptional sets has probability one, as has been mentioned above. Therefore, the main emphasis of the law is on the fact that for large c the lower density is bounded away from zero by a strictly positive constant depending only on c.

The relation of the Hölder continuity to the square root law can be seen from the fact that if u_t satisfies

$$du_t = (1/2)D^2 u_t \, dt + \sigma D u_t \, dw_t \quad \text{in} \quad x > 0, t > 0,$$

where σ is a constant, then the function

$$v_t(x) = u_t(x - \sigma w_t)$$

satisfies the deterministic heat equation

$$\frac{\partial v_t}{\partial t} = (1/2)(1-\sigma^2)D^2 u_t$$

in the random domain

$$x > \sigma w_t, \quad t > 0.$$

Knowing the self-similar behavior of the boundary helps proving the Hölder continuity of v and u in x.

Bibliography

1. B. Davis, *On Brownian slow points*, Z. Wahrscheinlichkeitstheory verw. Gebiete **64** (1983), 359–367.
2. T. Funaki, *Random motion of strings and related evolution equations*, Nagoya Math. Journal **89** (1983), 129–193.
3. N. Konno and T. Shiga, *Stochastic partial differential equations for some measure–valued diffusions*, Probab. Theory Relat. Fields **79** (1988), 201–225.
4. N.V. Krylov, *A W_2^n-theory of the Dirichlet problem for SPDE in general smooth domains*, Probab. Theory Relat Fields **98** (1994), 389–421.
5. N.V. Krylov, *A generalization of the Littlewood-Paley inequality and some other results related to stochastic partial differential equations*, Ulam Quarterly **2** (1994), no. 4, 16–26, http://www.ulam.usm.edu/VIEW2.4/krylov.ps.
6. N.V. Krylov, *Elements of an L_p–theory of SPDEs in the whole space*, 12 pages in "Proceedings of an International Workshop on Stochastic Partial Differential Equations", G. Kallianpur, M.R. Leadbetter org., June 19-24, 1994, Center For Stochastic Processes, University of North Carolina, Chapel Hill, 1994.
7. N.V. Krylov, *On L_p-theory of stochastic partial differential equations in the whole space*, SIAM J. Math. Anal. **27** (1996), no. 2, 313–340.
8. N.V. Krylov, *On SPDEs and superdiffusions*, Annals of Probability **25** (1997), no. 4, 1789–1809.
9. N.V. Krylov, *On a result of C. Mueller and E. Perkins*, Probab. Theory Relat. Fields **108** (1997), no. 4, 543–557.
10. N.V. Krylov, *An analytic approach to SPDEs*, pp. 185-242 in "Stochastic Partial Differential Equations: Six Perspectives", Mathematical Surveys and Monographs, Vol. 64, AMS, Providence, RI, 1999..
11. N.V. Krylov, *Constructing the super–Brownian process by using SPDEs and Skorohod's method*, pp. 232-240 in "Skorokhod's Ideas in Probability Theory", Proceedings of the Institute of Math. of the Nat. Acad, of Sci. of Ukraine, V. Korolyuk, N. Portenko, H. Syta eds, Vol. 32, Institute of Math., Kyiv, 2000..
12. N.V. Krylov, *One more square root law for Brownian motion and its application to SPDEs*, Probab. Theory Relat. Fields **127** (2003), 496–512.
13. N.V. Krylov, *On the foundation of the L_p-theory of SPDEs*, pp. 179-191 in "Stochastic Partial Differential Equations and Applications-VII", G. Da Prato, L. Tubaro eds., A Series of Lecture Notes in Pure and Applied Math., Chapman & Hall/CRC, 2006.
14. N.V. Krylov, *Maximum principle for SPDEs and its applications*, pp. 311-338 in "Stochastic Differential Equations: Theory and Applications, A Volume in Honor of Professor Boris L. Rozovskii", P.H. Baxendale, S.V. Lototsky eds., Interdisciplinary Mathematical Sciences, Vol. 2, World Scientific, 2007.
15. C. Mueller, *Long time existence for the heat equation with a noise term*, Probab. Theory Related Fields **90** (1991), 505–517.
16. C. Mueller and E. Perkins, *The compact support property for solutions to the heat equation with noise*, Probab. Theory Relat. Fields **93** (1992), 325–358.
17. M. Reimers, *One dimensional stochastic partial differential equations and the branching measure diffusion*, Probab. Theory Relat. Fields **81** (1989), 319–340.
18. J. B. Walsh, *An introduction to stochastic partial differential equations*, Lecture Notes in Math. **1180** (1986), 266–439.

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