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PAVLO KONONCHUK

PASTING OF TWO DIFFUSION PROCESSES ON A LINE WITH NONLOCAL BOUNDARY CONDITIONS

In this paper, we obtain an integral representation of an operator semigroup that describes the Feller process on a line that is a result of pasting together two diffusion processes with a nonlocal condition of conjugation.

1. Introduction and formulation of the problem. Let $\mathcal{D}_1 = \{x \in \mathbb{R}^1 : x < 0\}$ and $\mathcal{D}_2 = \{x \in \mathbb{R}^1 : x > 0\}$ be domains on $\mathbb{R}^1$, $S = \{0\}$ is boundary of the domain $\mathcal{D}_i$, $\overline{\mathcal{D}}_i = \mathcal{D}_i \cup \{0\}$ is the closure of $\mathcal{D}_i$, $i = 1, 2$. Assume that $\mathcal{L}_i$ is a second-order differential operator that operates on the set $\mathcal{C}^2_K(\overline{\mathcal{D}}_i)$ of all twice continuously differentiable functions with compact supports:

$$\mathcal{L}_i\varphi(x) = \frac{1}{2}b_i(x)\frac{d^2\varphi(x)}{dx^2} + a_i(x)\frac{d\varphi(x)}{dx}, \tag{1}$$

where $b_i(x)$ and $a_i(x), i = 1, 2$, are continuous and bounded on $\overline{\mathcal{D}}_i$, and also $b_i(x) \geq 0$. Let us also assume that, at the point $x = 0$, the following integral operator is defined:

$$\mathcal{L}_0\varphi(0) = \int_{\mathbb{R}^1} (\varphi(0) - \varphi(y))\, \mu(dy), \tag{2}$$

where $\mu(\cdot)$ is a nonnegative Borel measure on $\mathbb{R}^1$, $\mu(\mathbb{R}^1) > 0$.

We assume that operator (2) is a part of the general Feller–Wentzell boundary operator ([1], [2]) that corresponds to jumps of the process after its reach to the boundary S.

Let us consider the following problem: to construct an operator semigroup $\mathcal{T}_t$ that generates the Feller process on $\mathbb{R}^1$ such that it coincides with a diffusion process controlled by the operator $\mathcal{L}_i$, $i = 1, 2$, and its behavior at the point $x = 0$ is determined by the boundary condition

$$\mathcal{L}_0\varphi(0) = 0. \tag{3}$$

The formulated problem is often called the problem of pasting of diffusion processes on a line (e.g., see [3], [4]).

Note that the problem of pasting of two diffusion processes on a line was previously considered in the most general formulation in [5]. Also, the problem of existence of a Feller semigroup that describes a diffusion process on a domain with boundary conditions of kind (2) has been considered in [6]. In the papers mentioned above, solutions of respective problems were found, by using methods of functional analysis.

In this article, the desired semigroup will be constructed by analytical methods using a solution of the corresponding conjugation problem for a second-order parabolic linear

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equation with discontinuous coefficients. The problem is to find a function $u(t,x)$ ($t > 0, x \in \mathbb{R}^1$ that satisfies the following conditions:

$$\frac{\partial u}{\partial t} = \mathcal{L}_i u, \quad t > 0, x \in \mathcal{D}_1 \cup \mathcal{D}_2, \tag{4}$$

$$u(0,x) = \varphi(x), \quad x \in \mathcal{D}_1 \cup \mathcal{D}_2, \tag{5}$$

$$u(t,-0) = u(t,+0), \quad t > 0, \tag{6}$$

$$\int_{\mathbb{R}^1} (u(t,0) - u(t,y))\,\mu(dy) = 0, \quad t > 0. \tag{7}$$

Note that, in problem (4)–(7), relationship (6) represents the Feller property for the process, and equality (7) represents boundary condition (3).

For solving problem (4)–(7), we will use the classical potential method. This approach allows us to obtain an integral representation for the concerned semigroup.

2. Basic notations. Let $\mathcal{D}_t^r$ and $\mathcal{D}_x^p$ be symbols of partial derivative with respect to t of order r and with respect to x of order p, respectively, where r and p are nonnegative integers; $\mathcal{B}(\mathbb{R}^1)$ is the Banach space of all measurable bounded real-valued functions on $\mathbb{R}^1$ with the norm $\|\varphi\| = \sup_x |\varphi(x)|$; T is a fixed number; $\mathbb{R}^2_\infty = (0,\infty) \times \mathbb{R}^1$; $\mathbb{R}^2_T = (0,T) \times \mathbb{R}^1$; Ω is a domain in $\mathbb{R}^2_\infty$ or in $\mathbb{R}^2_T$; $\mathcal{C}(\Omega)$ ($\mathcal{C}(\overline{\Omega})$) is a set of functions continuous in Ω ($\overline{\Omega}$) with continuous derivatives D_t, D_x^p, $p = 1,2$ on Ω ($\overline{\Omega}$); $H^\alpha(\mathbb{R}^1)$, $\alpha \in (0,1)$, denotes the Hölder space as in [7, p.16]; and $\mathcal{D}_\delta = \{x \in \mathbb{R}^1 : |x| > \delta > 0\}$; C, c are positive constants not depending on (t,x), whose exact values are irrelevant.

3. Solution of the parabolic conjugation problem using analytical methods. Additionally, we assume that $\mathcal{L}_1$, $\mathcal{L}_2$ from (1) and the measure $\mu(\cdot)$ from (2) satisfy the following conditions:

a) functions $b_i(x)$, $a_i(x)$, $i = 1,2$, are defined on $\mathbb{R}^1$, and $b_i, a_i \in H^\alpha(\mathbb{R}^1)$;

b) there exist constants b_{0i}, b_{1i}, $i = 1,2$, such that $0 < b_{0i} \le b_{1i}$, $i = 1,2$, and $b_{0i} \le b_i(x) \le b_{1i}$, $i = 1,2$;

c) there exists $\Delta > 0$ such that, for $0 < \delta < \Delta$ and for all functions φ from $\mathcal{B}(\mathbb{R}^1)$,

$$\left| \int_{\mathcal{D}_\delta} \varphi(y)\mu(dy) \right| \le C_1 \|\varphi\|, \tag{8}$$

$$\left| \int_{\mathbb{R}^1 \setminus \mathcal{D}_\delta} \varphi(y)\mu(dy) \right| \le C_2(\delta) \|\varphi\|, \tag{9}$$

where $C_1 > 0$ does not depend on δ, and $C_2(\delta) \to 0$ as $\delta \to 0$.

Remark 1. From a) and b), it follows (see [7]) that there exists a fundamential solution (f.s.) of Eq. (4) that will be denoted by $g_i(t,x,y)$ ($t > 0$, $x, y \in \mathbb{R}^1$), $i = 1,2$.

Remark 2. c) implies that $\mu\{0\} = 0$ and $\mu(\mathbb{R}^1) < \infty$. Without loss of generality, we consider $\mu(\mathbb{R}^1) = 1$.

Let us recall some known properties of f.s. g_i, $i = 1,2$ that we will use further in our paper:

1) the function $g_i(t,x,y)$ is nonnegative continuous in all variables and is expressed by the formula

$$g_i(t,x,y) = g_{i0}(t,x,y) + g_{i1}(t,x,y), \quad t > 0, x, y \in \mathbb{R}^1, i = 1,2, \tag{10}$$

where

$$g_{i0} = (2\pi b_i(y)t)^{-\frac{1}{2}} \exp\left(-\frac{(x-y)^2}{2b_i(y)t}\right),$$

$g_{i1}(t,x,y)$ is in the form of an integral operator with a kernel g_{i0} and a density Φ_{i0} that is defined from some integral equation ($g_{i1} \equiv 0$ when $t \leq 0$);

2) the function $g_i(t,x,y)$, $i = 1,2$, as a function of arguments t and x, is continuously differentiable with respect to t and twice continuously differentiable with respect to x, and

$$|D_t^r D_x^p g_i(t,x,y)| \leq Ct^{-\frac{1+2r+p}{2}} \exp\left(-c\frac{|x-y|^2}{t}\right), \tag{11}$$

$$|D_t^r D_x^p g_{i1}(t,x,y)| \leq Ct^{-\frac{1+2r+p-\alpha}{2}} \exp\left(-c\frac{|x-y|^2}{t}\right), \tag{12}$$

where $2r+p \leq 2$, $0 < t \leq T$.

We now establish a classical solvability of problem (4)–(7) on the space of all functions continuous and bounded in the variable x.

Theorem 1. *Assume that conditions a)-c) hold for the coefficients of the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ from (1) and for the measure $\mu(\cdot)$ from (2). Then, for every continuous function $\varphi \in \mathcal{B}(\mathbb{R}^1)$ from (5), problem (4)–(7) has a unique solution*

$$u \in \mathcal{C}^{1,2}((0,\infty)\times\mathcal{D}_i) \cap \mathcal{C}((0,\infty)\times\mathbb{R}^1), \quad i = 1,2, \tag{13}$$

for which ($t \in (0,T]$, $x \in \mathbb{R}^1$)

$$|u(t,x)| \leq C\|\varphi\|, \tag{14}$$

and this solution can be obtained in the form

$$u(t,x) = \int_{\mathbb{R}^1} g_i(t,x,y)\varphi(y)dy + \int_0^t g_i(t-\tau,x,0)V_i(\tau)d\tau, \quad t>0, x\in\mathcal{D}_i, i=1,2, \tag{15}$$

where $V_i(t)$ ($t > 0$) is a solution of some system of second-kind Volterra integral equations.

Proof. According to the statement of Theorem 1, we will find a solution of problem (4)–(7) as (15), where V_i, $i = 1,2$, are the unknown functions that will be defined from the conjugation conditions (6) and (7). To this end, we denote the first and second terms on the right-hand side of Eq. (15) by $u_{i0}(t,x)$ and $u_{i1}(t,x)$ $i = 1,2$, respectively. Substituting the expression for $u(t,x)$ in conditions (6),(7), we obtain a system of equations for V_i, $i = 1,2$,

$$\int_0^t g_i(t-\tau,0,0)V_i(\tau)d\tau - \sum_{j=1}^{2}\int_0^t \left(\int_{\mathcal{D}_j} g_j(t-\tau,y,0)\mu(dy)\right) V_j(\tau)d\tau = \Phi_i(t), \quad t>0, \qquad i=1,2, \tag{16}$$

where

$$\Phi_i(t) = \sum_{j=1}^{2}\int_{\mathcal{D}_j} u_{j0}(t,y)\mu(dy) - u_{i0}(t,0), \quad i=1,2.$$

One can see that the system of equations (16) is a system of first-kind Volterra integral equations. Using the Holmgren hold (e.g., see [8]), we transform it to an equivalent system of second-kind Volterra integral equations. For this, we define the operator

$$\mathcal{E}(t)\Phi = \sqrt{\frac{2}{\pi}}\frac{d}{dt}\int_0^t (t-s)^{-\frac{1}{2}}\Phi(s)ds, \quad t>0,$$

and apply it to both sides of the system of equations (16). Taking into account properties 1), 2) for g_i and condition c), we obtain, after easy reductions, the equalities

$$V_i(t) = \sum_{j=1}^{2} \int_0^t K_{ij}(t-\tau)V_j(\tau)d\tau + \Psi_i(t), \quad t > 0, i = 1,2, \tag{17}$$

where

$$\begin{aligned}
K_{ii}(t-\tau) &= \sqrt{\frac{2b_i(0)}{\pi}}\frac{d}{dt}\int_\tau^t (t-s)^{-\frac{1}{2}}\left(\int_{\mathcal{D}_i} g_{i1}(s-\tau,y,0)\mu(dy) - g_{i1}(s-\tau,0,0)\right)ds- \\
&\quad - b_i(0)\int_{\mathcal{D}_i}\frac{\partial g_{i0}(t-\tau,y,0)}{\partial y}\mu(dy), \quad i=1,2, \\
K_{ij}(t-\tau) &= \sqrt{\frac{2b_i(0)}{\pi}}\frac{d}{dt}\int_\tau^t (t-s)^{-\frac{1}{2}}\left(\int_{\mathcal{D}_j} g_{j1}(s-\tau,y,0)\mu(dy)\right)ds \\
&\quad - \sqrt{b_i(0)b_j(0)}\int_{\mathcal{D}_j}\frac{\partial g_{j0}(t-\tau,y,0)}{\partial y}\mu(dy), \quad i,j=1,2, \\
\Psi_i(t) &= \sqrt{b_i(0)}\mathcal{E}(t)\Phi_i, \quad i=1,2, i\neq j.
\end{aligned}$$

Equations (17) form a system of second-kind Volterra integral equations for V_i, $i = 1,2$. In this system of equations, the first terms that are a part of expressions for kernels $K_{ij}(t-\tau)$ (we will denote them $K_{ij}^{(1)}(t-\tau)$) and $\Psi_i(t)$ can be estimated as

$$\left|K_{ij}^{(1)}(t-\tau)\right| \le \sqrt{b_i(0)}C_T(t-\tau)^{-1+\frac{\alpha}{2}}, \quad i,j=1,2, \tag{18}$$

$$|\Psi(t)| \le \sqrt{b_i(0)}K_T\|\varphi\|t^{-\frac{1}{2}}, \quad i=1,2, \tag{19}$$

which holds in every domain of the form $0 \le \tau < t \le T$ and $0 < t \le T$, respectively, with some constants C_T and K_T. Estimations (18) and (19) have similar proofs. For example, we will prove estimation (19). By differentiating the integral in the expression for $\Psi_i(t)$, we obtain the formula

$$\Psi_i(t) = \sqrt{\frac{2b_i(0)}{\pi}}\left(\frac{1}{2}\int_0^t (t-s)^{-\frac{3}{2}}(\Phi_i(t)-\Phi_i(s))ds + t^{-\frac{1}{2}}\Phi_i(t)\right), \quad i=1,2. \tag{20}$$

After that, estimating the right-hand side of (20), using inequality (11), and the formula of finite growth for the difference $\Phi_i(t) - \Phi_i(s)$, we obtain ($0 < t \le T$)

$$|\Phi_i(t)| \le C\|\varphi\|, \quad |\Phi_i(t)-\Phi_i(s)| \le C\|\varphi\|s^{-1}(t-s),$$

thus

$$\begin{aligned}
|\Psi_i(t)| &\le \sqrt{b_i(0)}C\|\varphi\|\left(\int_0^{\frac{t}{2}}(t-s)^{-\frac{3}{2}}ds + \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}}s^{-1}ds + t^{-\frac{1}{2}}\right) \\
&\le \sqrt{b_i(0)}K_T\|\varphi\|t^{-\frac{1}{2}}.
\end{aligned}$$

Similarly we obtain estimation (18). Concerning the functions that determine the second term in the expression for $K_{ij}(t-\tau)$ (denote them by $K_{ij}^{(2)}$), the direct application of inequality (11) to the derivatives $\frac{\partial g_{j0}}{\partial y}$, $j = 1,2$, results in a nonintegrable singularity for these functions at $t = \tau$. Despite that, we will prove that the general method of successive approximations is applicable to the system of equations (17). This means that we may try to find s solution of the system of equations (17) in the form of a series

$$V_i(t) = \sum_{k=0}^{\infty} V_i^{(k)}(t), \quad i = 1, 2, \tag{21}$$

where

$$V_i^{(0)}(t) = \Psi_i(t),$$
$$V_i^{(k)}(t) = \sum_{j=1}^{2} \int_0 K_{ij}(t-\tau) V_j^{(k-1)}(\tau) d\tau, \quad k = 1, 2, \ldots$$

Estimate $V_i^{(1)}(t)$. For this, we use the equality

$$V_i^{(1)} = \sum_{l,j=1}^{2} \int_0^t K_{ij}^{(l)}(t-\tau) V_j^{(0)}(\tau) d\tau = \sum_{l,j=1}^{2} V_{ij}^{(1l)}(t), \quad i = 1, 2. \tag{22}$$

Taking into account (18) and (19), we obtain

$$\begin{aligned} \left|V_{ij}^{(11)}(t)\right| &\leq K_T \|\varphi\| \sqrt{b_j(0) b_i(0)} C_T \int_0^t (t-\tau)^{-1+\frac{\alpha}{2}} \tau^{-\frac{1}{2}} d\tau \\ &= K_T \|\varphi\| t^{-\frac{1}{2}} \sqrt{b_i(0) b_j(0)} \frac{C_T \Gamma(\frac{\alpha}{2}) \Gamma(\frac{1}{2})}{\Gamma(\frac{1+\alpha}{2})} t^{\frac{\alpha}{2}}. \end{aligned} \tag{23}$$

Before estimating $V_{ij}^{(12)}$, we recall the formula

$$V_{ij}^{(12)}(t) = \sqrt{\frac{b_i(0)}{b_j(0)}} \frac{1}{\sqrt{2\pi b_j(0)}} \int_{\mathcal{D}_j} \mu(dy) \int_0^t \frac{y}{(t-\tau)^{\frac{3}{2}}} \exp\left(-\frac{y^2}{2b_j(0)(t-\tau)}\right) V_j^{(0)}(\tau) d\tau,$$
$$i, j = 1, 2.$$

Using (19), we obtain

$$\begin{aligned} &\left|V_{ij}^{(12)}(t)\right| \\ &\quad \leq K_T \|\varphi\| \sqrt{b_i(0)} \int_{\mathcal{D}_j} \mu(dy) \frac{1}{\sqrt{2\pi b_j(0)}} \int_0^t \frac{|y|}{(t-\tau)^{\frac{3}{2}} \tau^{\frac{1}{2}}} \exp\left(-\frac{y^2}{2b_j(0)(t-\tau)}\right) d\tau. \end{aligned}$$

Since

$$\frac{1}{\sqrt{2\pi b_j(0)}} \int_0^t \frac{|y|}{(t-\tau)^{\frac{3}{2}} \tau^{\frac{1}{2}}} \exp\left(-\frac{y^2}{2b_j(0)(t-\tau)}\right) d\tau = t^{-\frac{1}{2}} \exp\left(-\frac{y^2}{2b_j(0)t}\right),$$

we have

$$\left|V_{ij}^{(12)}(t)\right| \leq K_T \|\varphi\| \sqrt{b_i(0)} t^{-\frac{1}{2}} \int_{\mathcal{D}_j} \exp\left(-\frac{y^2}{2b_j(0)t}\right) \mu(dy). \tag{24}$$

From (22)–(24), we obtain the estimation

$$\begin{aligned} &\left|V_i^{(1)}(t)\right| \leq K_T \|\varphi\| \sqrt{b_i(0)} t^{-\frac{1}{2}} \\ &\quad \times \left(\frac{(\sqrt{b_1(0)} + \sqrt{b_2(0)}) C_T \Gamma(\frac{\alpha}{2}) \Gamma(\frac{1}{2})}{\Gamma(\frac{1+\alpha}{2})} t^{\frac{\alpha}{2}} + \sum_{j=1}^{2} \int_{\mathcal{D}_j} \exp\left(-\frac{y^2}{2b_j(0)t}\right) \mu(dy) \right), \\ &\qquad i = 1, 2, t \in (0, T]. \end{aligned} \tag{25}$$

On the right-hand side of (25), we use the notations

$$a_t = \frac{\left(\sqrt{b_1(0)}+\sqrt{b_2(0)}\right) C_T\Gamma(\frac{\alpha}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1+\alpha}{2})} t^{\frac{\alpha}{2}}, \quad b_t = \sum_{j=1}^{2}\int_{\mathcal{D}_j}\exp\left(-\frac{y^2}{2b_j(0)t}\right)\mu(dy).$$

Notice that condition c) guarantees the following inequality for b_t:

$$b_t \le b_T \le \int_{\mathbb{R}^1}\exp\left(-\frac{y^2}{2b(0)T}\right)\mu(dy) < 1,$$

where $b(0)=\max\{b_1(0), b_2(0)\}$.

Further, by using the method of induction on k, we establish the following estimation for $V_i^{(k)}(t)$:

$$\left|V_i^{(k)}(t)\right| \le K_T\|\varphi\|\sqrt{b_i(0)}t^{-\frac{1}{2}}\sum_{m=0}^{k} C_k^m a_t^{(k-m)} b_t^m, \quad i=1,2, k=0,1,2,\dots, \tag{26}$$

where

$$a_t^{(m)} = \frac{\left(\left(\sqrt{b_1(0)}+\sqrt{b_2(0)}\right) C_T\Gamma(\frac{\alpha}{2})\right)^m\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+m\frac{\alpha}{2})} t^{m\frac{\alpha}{2}}, \quad m=0,1,\dots,k.$$

Here, C_T and K_T are the constants from inequalities (18) and (19), respectively. Taking estimation (26) into account, we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\left|V_i^{(k)}(t)\right| &\le K_T\|\varphi\|\sqrt{b_i(0)}t^{-\frac{1}{2}}\sum_{k=0}^{\infty}\sum_{m=0}^{k} C_k^m a_t^{(k-m)} b_t^m \\ &= K_T\|\varphi\|\sqrt{b_i(0)}t^{-\frac{1}{2}}\sum_{k=0}^{\infty} a_t^{(k)}\sum_{m=0}^{\infty} C_{k+m}^m b_t^m \\ &= K_T\|\varphi\|\sqrt{b_i(0)}t^{-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{a_t^{(k)}}{(1-b_t)^{k+1}} \\ &\le K_T\|\varphi\|\sqrt{b_i(0)}t^{-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{\left((\sqrt{b_1(0)}+\sqrt{b_2(0)}C_T\Gamma(\frac{\alpha}{2})\right)^k\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+k\frac{\alpha}{2})(1-b_t)^{k+1}}.\end{aligned}\tag{27}$$

Inequality (27) guarantees the convergence of the series in (21) and gives the estimation for V_i, $i=1,2$:

$$|V_i(t)| \le C\|\varphi\|t^{-\frac{1}{2}} \quad i=1,2, \tag{28}$$

where $t\in(0,T]$, and C is some constant.

So we have constructed a system of integral equations (17) and verified an estimation for (28). The given estimation and (12) for $r=p=0$ ensure the existence of the function u_{i1}, $i=1,2$ from (15) and inequality (14) for it. It is obvious that the same inequality holds for the function u_{i0}, $i=1,2$, from (15) that holds also for the function u. This means that we have proven the existence of a solution of problem (4)–(7).

Remark 3. If we add condition (3) to the statements of Theorem 1, then the obtained solution of problem (4)–(7) belongs to the space $\mathcal{C}([0,\infty)\times\mathbb{R}^1)$.

Let us prove the uniqueness of the constructed solution of problem (4)–(7). Assume that there exist two distinct solutions of the problem that belong to class (13). We denote them by $u^{(1)}(t,x)$ and $u^{(2)}(t,x)$. Then the function $u(t,x)=u^{(1)}(t,x)-u^{(2)}(t,x)$ is a solution of problem (4)–(7) when $\varphi(x)\equiv 0$ which is continuous in the domain $[0,\infty)\times\mathbb{R}^1$, so its parts in the domains $(t,x) \in [0,\infty) \times \overline{\mathcal{D}_1}$ and $(t,x) \in [0,\infty) \times \overline{\mathcal{D}_2}$ are, at the same time, solutions of the first parabolic boundary-value problem

$$D_t u = \mathcal{L}_i u, \quad (t,x) \in (0,\infty) \times \mathcal{D}_i, i = 1,2, \tag{29}$$

$$u(0,x) = 0, \quad x \in \overline{\mathcal{D}_i}, i = 1,2, \tag{30}$$

$$u(t,0) = v(t), t \geq 0 \tag{31}$$

where

$$v(t) = \int_{\mathbb{R}^1} \left(u^{(1)}(t,y) - u^{(2)}(t,y) \right) \mu(dy).$$

Since the function $v(t)$ has the Hölder property when $t > 0$, the first boundary-value problem has a unique solution that can be represented as

$$u(t,x) = \int_0^t g_i(t-\tau,x,0) V_i(\tau) d\tau, \quad (t,x) \in (0,\infty) \times \mathcal{D}_i, i = 1,2 \tag{32}$$

(e.g., see [8]). Following the proof of the existence of a solution of problem (4)–(7) given above, one can notice that the functions $V_1(t)$ and $V_2(t)$ from (32) are, at the same time, the unique solutions of the homogeneous system of integral equations (17), where $\Psi_i(t) \equiv 0$, $i = 1,2$. So $V_i(t) = 0$ $(i = 1,2)$, which yields $u(t,x) \equiv 0$ and $u^{(1)}(t,x) \equiv u^{(2)}(t,x)$. Theorem 1 is proved.

4. Construction of a diffusion process. From Theorem 1, it follows that, using the solution of problem (4)–(7), we can determine the family of linear operators $(\mathcal{T}_t)_{t>0}$ that acts in the space $\mathcal{B}(\mathbb{R}^1)$. For $\varphi \in \mathcal{B}(\mathbb{R}^1)$, we put

$$\mathcal{T}_t \varphi(x) = \int_{\mathbb{R}^1} g_i(t,x,y)\varphi(y)dy + \int_0^t g_i(t-\tau,x,0) V_i(\tau,\varphi) d\tau, \quad t > 0, x \in \mathcal{D}_i, i = 1,2, \tag{32}$$

where $V_i(t,\varphi) \equiv V_i(t)$, $i = 1,2$, is a solution of the system of integral equations (17).

We will study properties of the operators $\{\mathcal{T}_t\}$ considering them on the space

$$\mathcal{M} = \left\{ \varphi \in \mathcal{B}(\mathbb{R}^1) \cap \mathcal{C}(\mathbb{R}^1) : \varphi(0) = \int_{\mathbb{R}^1} \varphi(y)\mu(dy) \right\}.$$

This restriction is related to the facts that we are firstly interested in Feller processes that are generated by the operators $\{\mathcal{T}_t\}$ and, secondly, one can assert that $\lim_{t\to 0} T_t\varphi(x) = \varphi(x)$ for every $x \in \mathbb{R}^1$, as it follows from Remark 3 [with fitting condition (3)]. One can easily prove that $\mathcal{M}$ is a closed subspace of the space of all bounded continuous functions on $\mathbb{R}^1$, and the operators $\{\mathcal{T}_t\}$ leave $\mathcal{M}$ invariant (that is, $\mathcal{T}_t\mathcal{M} \subset \mathcal{M}$ for every $t \geq 0$). We will show that the operators $\{\mathcal{T}_t\}$, $t \geq 0$ satisfy the following conditions:

1') if $\varphi_n \in \mathcal{M}$ when $n = 1,2,...$, $\sup_n \|\varphi_n\| < \infty$, and, for all $x \in \mathbb{R}^1$, we have $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$, then, for all $t > 0$, $x \in \mathbb{R}^1$, the next relations are satisfied: $\lim_{n\to\infty} V_i(t,\varphi) = V_i(t,\varphi)$, $i = 1,2$, and $\lim_{n\to\infty} \mathcal{T}_t\varphi(x) = \mathcal{T}_t\varphi(x)$ (the last relation is obviously satisfied even when $t = 0$);

2') $\mathcal{T}_t\varphi(x) \geq 0$ for all $t \geq 0$, $x \in \mathbb{R}^1$, whenever the function $\varphi \in \mathcal{M}$ satisfies the property that $\varphi(x) \geq 0$ for all $x \in \mathbb{R}^1$;

3') for all $t_1 \geq 0$, $t_2 \geq 0$, the next relation holds:

$$\mathcal{T}_{t_1+t_2} = \mathcal{T}_{t_1}\mathcal{T}_{t_2};$$

4') $\|\mathcal{T}_t\| \leq 1$ for all $t \geq 0$.

Conditions 1')-4') can be easily verified. In particular, condition 1') is a corollary from properties of the solution of the system of equations (17) (in series (21) that represent the function $V_i(t,\varphi_n)$, $i = 1,2$, we can take limit term-by-term) and from the Lebesgue theorem about passing to the limit under the sign of integral. Property 2') which means that the operator $\mathcal{T}_t$ leaves the cone of nonnegative functions from the space $\mathcal{M}$ to be invariant and property 3') called a semigroup property are corollaries of the maximum principle for parabolic equations ([9]) and the statement of Theorem 1 concerning the uniqueness of a solution of problem (4)–(7), respectively. Finally, to verify property 4') which means that, for every $t \geq 0$, the operator $\mathcal{T}_t$ is a contraction operator, it is enough to notice (remembering 2')) that $\mathcal{T}_t\varphi_0(x) \equiv 1$ for all $t \geq 0$, $x \in \mathbb{R}^1$, if only $\varphi_0(x) \equiv 1$.

Hence, we make conclusion (e.g., see [4]) that the operator semigroup $\mathcal{T}_t$, $t \geq 0$, constructed by formulas (32) and (17) determines some homogeneous Feller process on $\mathbb{R}^1$. Denote its transition probability by $\mathcal{P}(t, x, dy)$, so that

$$\mathcal{T}_t\varphi(x) = \int_{\mathbb{R}^1} \mathcal{P}(t, x, dy)\varphi(y).$$

Therefore, we have proved the next theorem.

Theorem 2. *Let the coefficients of the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ from (1) and the measure $\mu(\cdot)$ from (2) satisfy conditions a)-c). Then the solution of problem (4)–(7) constructed in Theorem 1 uniquely determines the operator semigroup $\mathcal{T}_t$, $t \geq 0$ that describes a homogeneous Feller process on $\mathbb{R}^1$ such that its parts at the inner points of the domains $\mathcal{D}_1$ and $\mathcal{D}_2$ coincide with the diffusion processes generated by the operators $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, and its behavior at $\{0\}$ is determined by the nonlocal conjugation condition (3).*

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Ukraine, 79000, Lviv, 1 Universytetska Str., Mechanics and Mathematics Faculty, Higher Mathematics Dept.

E-mail: p.kononchuk@gmail.com