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ON THE MARTINGALE PROBLEM FOR PSEUDO-DIFFERENTIAL OPERATORS OF VARIABLE ORDER

Consider parabolic pseudo-differential operators $L = \partial_t - p(x, D_x)$ of variable order $\alpha(x) \leq 2$. The function $\alpha(x)$ is assumed to be smooth, but the symbol $p(x, \xi)$ is not always differentiable with respect to x. We will show the uniqueness of Markov processes with the generator L. The essential point in our study is to obtain the L^p-estimate for resolvent operators associated with solutions to the martingale problem for L. We will show that, by making use of the theory of pseudo-differential operators and a generalized Calderon–Zygmund inequality for singular integrals. As a consequence of our study, the Markov process with the generator L is constructed and characterized. The Markov process may be called a stable-like process with perturbation.

1. Introduction and Notation

Set $D_x = -i\partial_x$, where $x = (x_j) \in \mathbf{R}^d$ and $\partial = \partial_x = (\partial/\partial_{x_j}))$. Then a symbol $p(x, \xi)$ is associated with the pseudo-differential operator $p(x, D_x)$ by the relation $p(x, D_x)e^{ix\cdot\xi} = e^{ix\cdot\xi}p(x, \xi)$. We consider a symbol

$$-p(x, \xi) \equiv \psi(x, \alpha(x), \xi) + \varphi(x, \xi)$$

which is a negative definite function of ξ, where α, ψ, φ are functions satisfying the following condition.

(1) $\mathbf{R}^d \ni x \longrightarrow \alpha(x) \in (0, 2]$: smooth,

(2) $\psi(x, \gamma, \lambda\xi) = \lambda^\gamma \ \psi(x, \gamma, \xi) \quad (\lambda > 0)$,

(3) $(0, 2] \ni \gamma \longrightarrow \psi(x, \gamma, \xi) \quad (|\xi| = 1)$: smooth,

(4) $x \longrightarrow {\partial_\xi}^\nu \psi(x, \gamma, \xi) \quad (|\nu| \leq d + 1)$: continuous and bounded,

(5) $\exists\varepsilon > 0, \ \varphi(x, \xi) = o(|\xi|^{\alpha(x)-\varepsilon}) \quad (|\xi| \to \infty)$.

Let $W = D(\mathbf{R}_+ \to \mathbf{R}^d)$ be the càd-làg path space, and $X_t(w) := w(t)$ for $w = (w(t)) \in W$. Set $\mathcal{W}_t = \bigcap_{\varepsilon>0} \sigma(X_s; s \leq t + \varepsilon)$, $\mathcal{W} = \sigma(X_s; s < \infty)$. We consider a parabolic pseudo-differential operator $L = \partial_t - p(x, D_x)$. A probability measure P on $(W, \mathcal{W})$ is called a solution to the martingale problem for the operator L if the process

$$\left(\exp[\ iX_t \cdot \xi + \int_0^t p(X_s, \xi)\ ds\ ]\right)$$

is a martingale w.r.t. $(\mathcal{W}_t, P)$ for any $\xi \in \mathbf{R}^d$. It is usually expected that the process $(W, (\mathcal{W}_t), P; X_t)$ is a Markov process with the generator L.

Bass [1] and Negoro [10] studied on a Markov process with the generator $-(-\Delta)^{\alpha(x)/2}$ of variable order $0 < \alpha(x) < 2$. The Markov processes associated with pseudo-differential operators with smooth symbols were studied in several articles (Hoh [3], Jacob-Leopold [4], Jacob [5], etc). There are two typical cases where the martingale problem for L is well-posed.

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Case 1: $p(x,\xi)$ *is a smooth symbol.* Applying the theory of pseudo-differential operators, under the non-degenerate condition

$$\sup \{\Re e\ \psi(x,\gamma,\xi) \mid x \in \mathbf{R}^d,\ 0<\gamma\le 2,\ |\xi|=1\} < 0,$$

we can show the existence of the smooth transition function of the Markov process with the generator L (Komatsu [8]).

Case 2: $\alpha(x)$ *is a constant function.* Using a generalized Hörmander inequality for singular integrals, under the non-degenerate condition, it is proved that the existence and the uniqueness of solutions to the martingale problem for the operator L hold good (Komatsu [6], Komatsu [7]).

One of the key points of this article is the unusual but well-devised definition of the pseudo-differential operator $\psi(x,\gamma,D_x)$, where the analytic distribution $\lambda \longrightarrow [r_+^\lambda]$ is used. Though the general notion of the analytic distribution can be found in [2], it might be better to give here a short sketch of the analytic distribution.

Let $\mathcal{D} = C_0^\infty(\mathbf{R}^d)$ be a space of test functions on $\mathbf{R}^d$, and let $\mathcal{D}'$ denote the space of distributions. Consider a distribution $f_\lambda = \langle f_\lambda,\ \cdot\ \rangle \in \mathcal{D}'$ with parameter $\lambda \in \Lambda$, where Λ is an open domain in $\mathbf{C}$. We say that f_λ is an analytic distribution if the function $\Lambda \ni \lambda \longrightarrow \langle f_\lambda, \phi\rangle$ is analytic for any $\phi \in \mathcal{D}$. Define derivatives $(d/d\lambda)^n f_\lambda$ by $\langle (d/d\lambda)^n f_\lambda, \phi\rangle = (d/d\lambda)^n \langle f_\lambda, \phi\rangle$. From the sequential completeness of the space $\mathcal{D}'$, we have $(d/d\lambda)^n f_\lambda \in \mathcal{D}'$, and the Taylor expansion

$$f_{\lambda+h} \ = \ \sum_{n=0}^{\infty} \frac{h^n}{n!}\ (\frac{d}{d\lambda})^n f_\lambda$$

holds in the sense of $\mathcal{D}'$. Then it is possible to consider the analytic continuation of the analytic distribution in the following way. Let f_λ $(\lambda \in \Lambda)$ be an analytic distribution, and $\Lambda \subset \Lambda_1 \subset \mathbf{C}$. Assume that the function $\Lambda \ni \lambda \longrightarrow \langle f_\lambda, \phi\rangle$ can be extended to the analytic function $\Lambda_1 \ni \lambda \longrightarrow g_\lambda(\phi)$ for any $\phi \in \mathcal{D}$, and set $\langle f_\lambda, \phi\rangle := g_\lambda(\phi)$. Then the distribution $\Lambda_1 \ni \lambda \longrightarrow f_\lambda$ is an analytic extension of the distribution $\Lambda \ni \lambda \longrightarrow f_\lambda$.

Let $d = 1$, and let $[x_\pm^\lambda]$ denote the analytic distribution defined for $\Re e\ \lambda > -1$ associated with the function $x_\pm^\lambda$ on $\mathbf{R}^1$. The largest extension of the analytic distribution $[x_\pm^\lambda]$ is the analytic distribution defined on $\Lambda = \{\lambda \in \mathbf{C} \mid \lambda \neq -1, -2, \cdots\}$. If $-n-1 < \Re e\ \lambda < -n$, the equality

$$\langle [x_\pm^\lambda], \phi(x)\rangle \ = \ \int_0^\infty x^\lambda \left(\phi(\pm x) - \sum_{k=0}^{n-1} \frac{(\pm x)^k}{k!}\phi^{(k)}(0)\right)\ dx$$

holds for any $\phi \in \mathcal{D}$. Note that the analytic distribution $\lambda \longrightarrow [x_\pm^\lambda]$ has poles of order 1 at negative integers, but its modification $\lambda \longrightarrow [x_\pm^\lambda]/\Gamma(\lambda+1)$ is an entire distribution. We see that

$$\lim_{\lambda\to -n} \langle\ \frac{[x_\pm^\lambda]}{\Gamma(\lambda+1)},\ \phi(x)\rangle \ = \ (\mp)^{n-1}\phi^{(n-1)}(0) \ = \ \langle \delta^{(n-1)}(x), \phi(x)\rangle.$$

On the other hand, the analytic distribution $\lambda \longrightarrow [(x \pm i0)^\lambda]$ is defined by

$$[(x\pm i0)^\lambda] \ = \ \begin{cases} [x_+^\lambda] + e^{\pm i\pi\lambda}[x_-^\lambda] & (-\lambda \notin \mathbf{N}), \\ [x^{-n}] \pm i\pi(-1)^n/(n-1)! \times \delta^{(n-1)}(x) & (-\lambda = n \in \mathbf{N}), \end{cases}$$

where the distribution $[x^{-n}]$ ($n \in \mathbf{N}$) is defined by the formula

$$\langle [x^{-n}], \phi(x)\rangle = \begin{cases} \int_0^\infty x^{-2m} \left(\phi(x) + \phi(-x) - 2\sum_{k=0}^{m-1} \frac{x^{2k}}{(2k)!}\phi^{(2k)}(0)\right) dx, & n = 2m, \\ \int_0^\infty x^{-2m-1} \left(\phi(x) - \phi(-x) - 2\sum_{k=0}^{m-1} \frac{x^{2k+1}}{(2k+1)!}\phi^{(2k+1)}(0)\right) dx, & n = 2m+1. \end{cases}$$

We have the equality

$$\mathcal{F}\left[\frac{x_\pm^{-\gamma-1}}{\Gamma(-\gamma)}\right](\xi) \ = \ \exp[\pm\frac{i\pi\gamma}{2}]\ (\xi \mp i0)^\gamma \qquad (\xi \in \mathbf{R}^1)$$

where $\mathcal{F}[\ \cdot\ ](\xi)$ denotes the Fourier transform in the distribution sense. Since the Fourier transform of an entire distribution is also an entire distribution, $\gamma \longrightarrow (\xi \mp i0)^\gamma$ is an entire distribution.

Consider the general case $d \geq 1$. Let $\sigma(d\omega)$ be the area element on S^{d-1}. It is natural to define the analytic distribition $\lambda \longrightarrow [|x|^\lambda]$ associated with the function $|x|^\lambda$ on $\mathbf{R}^d$ by

$$\langle [|x|^\lambda], \phi(x)\rangle \ = \ \langle [r_+^{\lambda+d-1}], \int_{|\omega|=1} \phi(r\omega)\ \sigma(d\omega)\rangle \qquad (\lambda + d \neq 0, -2, -4, \cdots),$$

because these equalities hold in the usual sense for $\Re e\ \lambda > -d$. This suggests a natural way to define the pseudo-differential operator $\psi(x,\gamma,D_x)$. Consider a function $m(x,\gamma,\omega)$ on $\mathbf{R}^d \times (0,2] \times S^{d-1}$ such that

(1) $\quad \forall (x,\gamma),\ m(x,\gamma,\omega) \in C^d(S^{d-1})$,

(2) $\quad (0,2] \ni \gamma \longrightarrow m(x,\gamma,\omega) \geq 0$: smooth,

(3) $\quad m(x,2,\omega) = 0, \qquad \int_{|\omega|=1} \omega\ m(x,1,\omega)\ \sigma(d\omega) = 0.$

We define a pseudo-differential operator $\psi(x,\gamma,D_x)$ of order $0 < \gamma \leq 2$ by

$$\psi(x,\gamma,D_x)f(x) = \langle [r_+^{-\gamma-1}], \int_{|\omega|=1} f(x+r\omega)\ m(x,\gamma,\omega)\sigma(d\omega)\rangle.$$

Note that the analytic distribution $\gamma \longrightarrow [r_+^{-\gamma-1}]$ has poles of order 1 at non-negative integers. For $\gamma = 1, 2$, we have

$$\psi(x,1,D_x)f(x) = \int (f(x+y) - f(x) - I_{\{|y|\leq 1\}}\ y\cdot\partial f(x))\ m(x,1,y/|y|)|y|^{-d-1}\ dy$$
$$- \left(\int_{|\omega|=1} \omega\ [\partial_\gamma m(x,\gamma,\omega)]_{\gamma=1}\ \sigma(d\omega)\right)\cdot \partial f(x),$$
$$\psi(x,2,D_x)f(x) = \frac{1}{2}\ tr[\ \left(\int_{|\omega|=1} \omega\omega^*\ [\partial_\gamma m(x,\gamma,\omega)]_{\gamma=2}\ \sigma(d\omega)\right) (\partial\partial^* f(x))\ ].$$

Consider an operator $\varphi(x,D_x)$ defined by

$$\varphi(x,D_x)f(x) = \int [f(x+y) - f(x) - I_{\{\alpha(x)>1+\varepsilon, |y|\leq 1\}}\ y\cdot\partial f(x)]N(x,dy)$$
$$+ I_{\{\alpha(x)>1+\varepsilon\}}\ b(x)\cdot\partial f(x).$$

We assume that the following condition is satisfied:

$$\sup_x |b(x)| + \int \sup_x (1 \wedge |y|^{\alpha(x)-\varepsilon})|N(x,dy)| < \infty.$$

Then we see that $\varphi(x,\xi) = o(|\xi|^{\alpha(x)-\varepsilon})$. It is not necessary that $b(\cdot)$ and $N(\cdot,dy)$ be continuous.

Theorem. *Under the non-degenerate condition that $\psi(x,\gamma,\xi)<0$ for $(x,\gamma,\xi)\in\mathbf{R}^d\times(0,2]\times S^{d-1}$, the martingale problem for the operator*

$$L=\partial_t+\psi(x,\alpha(x),D_x)+\varphi(x,D_x)$$

is well-posed, that is, the existence and the uniqueness of solutions holds good.

2. Estimates for Fundamental Solutions

One of the bases of our reasoning is the theory of pseudo-differential operators. For a bdd function $\zeta(x)$ and $0<\delta<1$, we define

$$|\widetilde{p}|_k^{\zeta}:=\sup_{|\beta+\gamma|\le k}\sup_{x,\xi}\{|\partial_\xi^\beta D_x^\gamma\widetilde{p}|\langle\xi\rangle^{|\beta|-\zeta(x)-\delta|\gamma|}\},$$

$$\mathcal{S}_{1,\delta}^{\zeta}=\{\widetilde{p}(x,\xi)\in C^\infty(\mathbf{R}^{2d})\mid|\widetilde{p}|_k^{\zeta}<\infty\ (\forall k)\},$$

where $\langle\xi\rangle=\sqrt{2+|\xi|^2}$. Each pseudo-differential operator in the class

$$\mathcal{P}_{1,\delta}^{\zeta}=\{\widetilde{p}(x,D_x)\mid\widetilde{p}(x,\xi)\in\mathcal{S}_{1,\delta}^{\zeta}\}$$

is called an operator of variable order $\zeta(x)$. If $p_j(x,\xi)\in\mathcal{S}_{1,\delta}^{\zeta_j}$ $(j=1,2)$, the symbol $(p_1\circ p_2)(x,\xi)$ of the iterated operator $p_1(x,D_x)p_2(x,D_x)$ belongs to the class $\mathcal{S}_{1,\delta}^{\zeta_1+\zeta_2}$, and the asymptotic expansion formula

$$p_1\circ p_2\ -\sum_{|\ell|<N}\frac{1}{\ell!}\,\partial_\xi^\ell p_1 D_x^\ell p_2\ \in\ \mathcal{S}^{\zeta_1+\zeta_2-N(1-\delta)}$$

holds for any $N\in\mathbf{Z}_+$ (see Kumano-go [9]).

Let $\rho(r)$ be a smooth function on $\mathbf{R}_+$ such that $\rho(r)=1$ for $r\le1$, $\rho(r)=0$ for $r\ge2$ and $0<\rho(r)<1$ for $1<r<2$. We fix a point $x_0\in\mathbf{R}^d$ and set

$$q(x,\xi)\ =\ -\ (\psi(x_0,\alpha(x),\cdot)*\hat{\rho})(\xi)\ +\ (\psi(x_0,\alpha(x),\cdot)*\hat{\rho})(0),$$

where $\hat{\rho}(\xi):=\mathcal{F}^{-1}[\rho(|\cdot|)](\xi)$. Note that we consider the symbol not $\psi(x_0,\alpha(x_0),\xi)$ but $\psi(x_0,\alpha(x),\xi)$. The symbol $q(x,\xi)$ belongs to the class $\mathcal{S}_{1,\delta}^{\alpha}$. Let $u(s,x,D_x)$ be the fundamental solution to the Cauchy problem for the operator $\partial_s+q(x,D_x)$.

We now survey how to construct the fundamental solution. Set $q\equiv q(x,\xi)$ and $u_0(s)\equiv u_0(s,x,\xi)=\exp(-sq)$. We may assume that there exists a constant $c>0$ such that

$$|u_0(s,x,\xi)|\le\exp(-cs\langle\xi\rangle^{\alpha(x)}).$$

Define symbols $\{u_j(s)\}_{j\ge1}$ by $u_j(0)=0$ and

$$-(\partial_s+q)u_j(s)=\sum_{|\ell|+k=j,|\ell|\ne0}\frac{1}{\ell!}\,\partial_\xi^\ell q\ D_x^\ell u_k(s).$$

The following estimates hold good (see [8], Lemma 3).

Proposition 1. *Fix $0<\delta<1$. There exist constants $C_{\beta\gamma j}$ such that*

$$\left|\frac{\partial_\xi^\beta D_x^\gamma u_j(s,x,\xi)}{u_0(s,x,\xi)}\right|\ \le\ C_{\beta\gamma j}\ \langle\xi\rangle^{-|\beta|+\delta|\gamma|-j(1-\delta)}\sum_{k=1}^{|\beta|+|\gamma|+2j}(s\langle\xi\rangle^{\alpha(x)})^k.$$

For sufficiently large N, we define the symbol

$$\widetilde{u}_N(s)\equiv\widetilde{u}_N(s,x,\xi):=\sum_{j=0}^{N-1}u_j(s).$$

From the asymptotic expansion formula, we have

$$\widetilde{r}_N(s) := -(\partial_s + q) \circ \widetilde{u}_N(s) \in \mathcal{S}_{1,\delta}^{\alpha(\cdot)-(1-\delta)N}.$$

The symbol $u(s) = u(s,x,\xi)$ of the fundamental solution can be constructed by

$$w_0(t) := \delta(t), \qquad w_j(s) = \int_0^s \widetilde{r}_N(\tau) \circ w_{j-1}(s-\tau) d\tau \quad (j \geq 1),$$

$$u(s,x,\xi) := \widetilde{u}_N(s,x,\xi) + \int_0^s \widetilde{u}_N(\tau) \circ (\sum_{j=1}^{\infty} w_j(s-\tau)) \ d\tau.$$

Hereafter, we assume that $\inf_x \alpha(x) > 0$. We define the resolvent operator G_λ $(\lambda > 0)$ by

$$G_\lambda f(x) = \int_0^\infty e^{-\lambda s} u(s,x,D_x) f(x) \ ds.$$

We use the convention of letting $c.$'s to stand for positive absolute constants. Each $c.$ may denote a constant different from other $c.$' s. From the next proposition and the Young inequality, we have the estimate

$$\lambda \|G_\lambda f\|_{L^p} \leq c. \ \|f\|_{L^p}.$$

Proposition 2. *For any $\beta \in \mathbf{R}^d$, there exists a constant c_β such that*

$$|\mathcal{F}^{-1}[\partial_\xi^\beta u(s,z,\xi)](y)| \ \leq \ c_\beta \qquad (s > 0, \ y, z \in \mathbf{R}^d),$$

and there is a constant C such that

$$\int_0^\infty e^{-s\lambda} \| \sup_z |\mathcal{F}^{-1}[u(s,z,\xi)](\cdot)| \ \|_{L^1} \ ds \ \leq \ C \ \frac{1}{\lambda}.$$

Proof. From Proposition 1, for $0 < s < 1$,

$$\begin{aligned} |\mathcal{F}^{-1}[\partial_\xi^\beta u_j(s,z,\xi)](y)| \ &= \ |\mathcal{F}^{-1}[\partial_\eta^\beta u_j(s,z,\eta)|_{\eta = s^{-1/\alpha}\xi}](s^{1/\alpha} y)| \\ &\leq \ c. \sum_{k=1}^{|\beta|+2j} \int (\langle s^{-1/\alpha}\xi \rangle^\alpha)^k \ e^{-sq(z,s^{-1/\alpha}\xi)} \ d\xi \\ &\leq \ c. + c. \int_{|\xi|>1} e^{-c.|\xi|^\alpha} \ d\xi \ \leq \ c. \ , \end{aligned}$$

where $\alpha = \alpha(z)$. It is much more easy to show that

$$\sup_{s \geq 1, y, z} \ |\mathcal{F}^{-1}[\partial_\xi^\beta u_j(s,z,\xi)](y)| \ < \ \infty.$$

These prove the first claim. The second claim is proved by the inequality

$$\langle y \rangle^{d+1} |\mathcal{F}^{-1}[u(s,z,\cdot)](y)| \leq c. \sum_{|\beta| \leq d+1} |\mathcal{F}^{-1}[\partial_\xi^\beta u(s,z,\xi)](y)|.$$

Similarly to the above proof, we can prove that there exist constants $c.$ such that

$$\begin{aligned} \langle y \rangle^d |\mathcal{F}^{-1}[u(s,z,\cdot)](s^{1/\alpha(z)} y)| \ &\leq \ c. \ s^{-d/\alpha(z)}, \\ \langle y \rangle^{d+1} |\mathcal{F}^{-1}[\xi_j u(s,z,\xi)](s^{1/\alpha(z)} y)| \ &\leq \ c. \ s^{-(d+1)/\alpha(z)}. \end{aligned}$$

Moreover, the following proposition can be proved with the use of Proposition 1 (see [7], Lemma 2.3).

Proposition 3. *Let $0 < \eta < \gamma \wedge 1$. There is a constant $C_{\eta\gamma}$ such that*

$$\langle y\rangle^{d+\eta}\ |\mathcal{F}^{-1}[\phi(\cdot)u(s,z,\cdot)](s^{1/\alpha(z)}y)|\ \leq\ C_{\eta\gamma}\left(\sup_{|\xi|=1}\sum_{|\beta|\leq d+1}|\partial_\xi^\beta\phi(\xi)|\right)\ s^{-(d+\gamma)/\alpha(z)}$$

for any homogeneous function $\phi(\xi)$ with index γ.

From the above-presented estimates and the Hölder inequality, we obtain the following proposition.

Proposition 4. *Let $\alpha_0 = \inf_x \alpha(x) > 0$.*
(1) *If $p\alpha_0 > d$, then $\|G_\lambda f\|_\infty \leq c.\ \lambda^{-1+d/p\alpha_0}\ \|f\|_{L^p}$,*
(2) *If $p(\alpha_0 - 1) > d$, then $\|D_x G_\lambda f\|_\infty \leq c.\ \lambda^{-1+(1+d/p)/\alpha_0}\ \|f\|_{L^p}$,*
(3) *If $0 < \eta < \alpha_0 \wedge 1$ and $(\alpha_0 - \eta)p > d$, then*

$$\|\ |D_x|^\eta G_\lambda f\ \|_\infty \leq c.\ \lambda^{-1+(\eta+d/p)/\alpha_0}\ \|f\|_{L^p}.$$

3. Estimates for Singular Integrals

Though the order function $\alpha(x)$ is smooth, the symbols $\psi(x,\alpha(x),\xi)$ and $p(x,\xi)$ are not smooth. Then we need the theory of singular integrals, as well as the theory of pseudo-differential operators, on which we will base the analysis for the operator $p(x,D_x)$. Let $\phi(\xi)$ be a homogeneous function with index 0, and let $\mu(\phi)$ be the average of $\phi(\cdot)$ over S^{d-1}. Then $k_\phi(x) := \mathcal{F}^{-1}[\phi](x) - \mu(\phi)\delta(x)$ is a homogeneous function with index $-d$. Define the singular integral operator $[f \longrightarrow k_\phi * f]$ by

$$(k_\phi * f)(x) = \lim_{\eta\downarrow 0}\int_{|y|>\eta} k_\phi(y)f(x-y)\ dy.$$

Then we have $\phi(D_x)f(x) = (k_\phi * f)(x) + \mu(\phi)f(x)$. The estimate in the following theorem (Komatsu [7], Theorem 2.1) is a key in this theory.

Lemma 1 (generalized Hörmander inequality).

$$\|\sup_z|\phi_z(D_x)f|\ \|_{L^p}\ \leq\ C_p\left(\sup_{z,|\xi|=1}\sum_{|\beta|\leq d}|\partial_\xi^\beta\phi_z(\xi)|\right)\|f\|_{L^p}$$

for any system $\{\phi_z(\xi)\}$ of homogeneous functions with index 0.

Define a pseudo-differential operator H by

$$Hf(x) = h(x,D_x)f(x) = \psi(x_0,\alpha(x),D_x)f(x) + q(x,D_x)f(x).$$

We have

$$h(x,\xi) = \langle[r_+^{-\alpha(x)-1}](1-\rho(r)),\ \int_{|\omega|=1} e^{ir\xi\cdot\omega}\ m(x_0,\alpha(x),\omega)\sigma(d\omega)\rangle + (\psi(x_0,\alpha(x),\cdot)*\hat{\rho})(0).$$

We see that the symbol $(1-\rho(|\xi|))(h(x,\xi)-h(x,0))$ belongs to the class $\mathcal{S}_{1,\delta}^{\alpha(x)-1}$.

Proposition 5. *Let $\{\psi_z(\gamma,\xi)\}$ be a system of functions on $(0,2]\times\mathbf{R}^d$ such that*
(1) $\psi_z(\gamma,\lambda\xi) = \lambda^\gamma\ \psi_z(\gamma,\xi)\quad(\lambda > 0)$,
(2) $(0,2]\times S^{d-1} \ni (\gamma,\xi) \longrightarrow \psi_z(\gamma,\xi)$ *is a smooth mapping.*
Then there exists a constant C_p such that

$$\|\ \sup_z|(\psi_z(\alpha(x),D_x)G_\lambda f)(x)|\ \|_{L^p}\ \leq\ C_p\left(\sup_{z,\gamma,|\xi|=1}\sum_{|\beta|\leq d+1}|\partial_\xi^\beta\psi_z(\gamma,\xi)|\right)\ \|f\|_{L^p}.$$

Proof. Set $\tilde{\phi}_z(\xi) := \psi_z(\gamma,\xi)/\psi(x_0,\gamma,\xi)$ which is independent of γ, and $\tilde{\rho}_0(\xi) := \rho(|\xi|)$, $\tilde{\rho}_1(\xi) := 1-\rho(|\xi|)$. Let $g_\lambda(x,\xi)$ be the symbol of a pseudo-differential operator G_λ:

$$g_\lambda(x,\xi) = \int_0^\infty e^{-\lambda s} u(s,x,\xi)\, ds.$$

Then we have

$$\begin{aligned}
&\psi_z(\alpha(x),\xi)\circ g_\lambda(x,\xi)\\
&= (\psi_z(\alpha(x),\xi)\tilde{\rho}_0(\xi))\circ g_\lambda(x,\xi) + (\psi_z(\alpha(x),\xi)\tilde{\rho}_1(\xi))\circ g_\lambda(x,\xi)\\
&= (\psi_z(\alpha(x),\xi)\tilde{\rho}_0(\xi))\circ g_\lambda(x,\xi) + (\tilde{\phi}_z(\xi)\tilde{\rho}_1(\xi)\psi(x_0,\alpha(x),\xi))\circ g_\lambda(x,\xi)\\
&= (\psi_z\tilde{\rho}_0)\circ g_\lambda + (\tilde{\phi}_z\tilde{\rho}_1 h)\circ g_\lambda - (\tilde{\phi}_z\tilde{\rho}_1 q)\circ g_\lambda\\
&= (\psi_z\tilde{\rho}_0)\circ g_\lambda + (\tilde{\phi}_z\tilde{\rho}_1 h)\circ g_\lambda + [(\tilde{\phi}_z\tilde{\rho}_1)\circ q - (\tilde{\phi}_z\tilde{\rho}_1)q]\circ g_\lambda - (\tilde{\phi}_z\tilde{\rho}_1)\circ q\circ g_\lambda\\
&= (\psi_z\tilde{\rho}_0)\circ g_\lambda + (\tilde{\phi}_z\tilde{\rho}_1 h)\circ g_\lambda + [(\tilde{\phi}_z\tilde{\rho}_1)\circ q - (\tilde{\phi}_z\tilde{\rho}_1)q]\circ g_\lambda + (\tilde{\phi}_z\tilde{\rho}_1)\circ(\lambda g_\lambda - 1).
\end{aligned}$$

Set $\eta = (\inf_x \alpha(x)\wedge 1)/2$ and

$$C_* = \sup_{z,\gamma,|\xi|=1} \sum_{|\beta|\le d+1} |\partial_\xi^\beta \psi_z(\gamma,\xi)|.$$

It can be proved that

$$\sup_{x,y,z} \langle y\rangle^{d+\eta} |\mathcal{F}^{-1}[\psi_z(\alpha(x),\cdot)\tilde{\rho}_0](y)| \ \le\ c.\ C_*.$$

We observe that the symbol $\tilde{p}_z(x,\xi) := [(\psi_z\tilde{\rho}_1)\circ q - \psi_z\tilde{\rho}_1 q)](x,\xi)$ belongs also to the class $\mathcal{S}_{1,\delta}^{\alpha(x)-1}$. We have estimates

$$\begin{aligned}
&\sup_y \langle y\rangle^{d+1}|\mathcal{F}^{-1}[(\tilde{\rho}_1 h)(x,\xi)\circ g_\lambda(x,\xi)](y)| \ \le\ c. \left(\frac{1}{\lambda}\right)^{(1/\alpha(x))\wedge 1},\\
&\sup_{y,z} \langle y\rangle^{d+1}|\mathcal{F}^{-1}[(\tilde{p}_z(x,\xi)\circ g_\lambda(x,\xi)](y)| \ \le\ c.\ C_* \left(\frac{1}{\lambda}\right)^{(1/\alpha(x))\wedge 1}.
\end{aligned}$$

It may not be a routine work to show these estimates, but these can be proved in a similar way to the proof of Proposition 2. Since

$$\begin{aligned}
\psi_z(\alpha(x),D_x)G_\lambda f(x) \ &= \psi_z(\alpha(x),D_x)\tilde{\rho}_0(D_x)G_\lambda f(x) + \tilde{\phi}_z(x,D_x)G_\lambda f(x)\\
&\quad + \tilde{\phi}_z(D_x)\tilde{\rho}_1(D_x)((h(x,D_x)+\lambda)G_\lambda f(x) - f(x)),
\end{aligned}$$

from the generalized Hörmander inequality and the Young inequality, the proof is completed.

Define the operators

$$\begin{aligned}
U_\lambda \ &=\ (q(x,D_x) - p(x,D_x))G_\lambda\\
&=\ (\psi(x,\alpha(x),D_x) - \psi(x_0,\alpha(x),D_x) + h(x,D_x) + \varphi(x,D_x))G_\lambda.
\end{aligned}$$

Since $\varphi(x,\xi) = o(|\xi|^{\alpha(x)-\varepsilon})$, it can be proved that

$$\|\varphi(x,D_x)G_\lambda\|_{L^p} \longrightarrow 0$$

as $\lambda\to\infty$ (see [6], Theorem 2). From Proposition 5, we see that U_λ is a bounded operator on L^p. Here, we assume that the value

$$\|\alpha(\cdot)-\alpha(x_0)\|_\infty + \sup_{\gamma,|\xi|=1} \sum_{|\beta|\le d+1} \|\partial_\xi^\beta\psi(\cdot,\gamma,\xi) - \partial_\xi^\beta\psi(x_0,\gamma,\xi)\|_\infty$$

is sufficiently small. Then there exists λ_0 such that $\|U_\lambda\|_{L^p} < 1$ for $\lambda \geq \lambda_0$. Let $p > d/(\inf_x \alpha(x))$, and let us define bounded operators on L^p by

$$R_\lambda = G_\lambda[I - U_\lambda]^{-1} \qquad (\lambda > \lambda_0).$$

We see from Proposition 4 that R_λ is a bounded operator from L^p to $C(\mathbf{R}^d) \bigcap L^p$ in both L^p and L^∞ norms. We have

$$(\lambda + p(x, D_x))R_\lambda f = f \qquad (f \in L^p),$$

and if $f \in [I - U_\lambda]C_0^\infty(\mathbf{R}^d)$, then $R_\lambda f \in G_\lambda\ C_0^\infty(\mathbf{R}^d) \subset C^\infty(\mathbf{R}^d)$.

4. L^p−Estimate and Proof of the Theorem

The proof of the uniqueness of solutions to the martingale problem is based on the following lemma (see [6], Lemma 3.1).

Lemma 2. *Let P^1 and P^2 be two probability measures on $(W, \mathcal{W})$ with $P^1[X_0 \in dx] = P^2[X_0 \in dx]$. Let $E^\ell[\ \cdot \mid \mathcal{W}_s]$ denote the conditional expectation by P^ℓ. The property*
$\forall s \geq 0,\ \forall \lambda \geq \lambda_0,\ \forall f \in C(\mathbf{R}^d) \cap L^p,\ \exists g \in C(\mathbf{R}^d),$

$$E^\ell \left[\int_0^\infty e^{-t\lambda} f(X_{s+t}) dt \mid \mathcal{W}_s \right] = g(X_s) \qquad (\ell = 1, 2)$$

implies that $P^1 = P^2$ on $\mathcal{W}$.

Let P be a solution to the martingale problem for $L = \partial_t - p(x, D_x)$. Then the process

$$M_t^\lambda := e^{-t\lambda} G_\lambda \phi(X_t) + \int_0^t e^{-s\lambda}(\lambda + p(x, D_x)) G_\lambda \phi(X_s)\ ds$$

is a martingale w.r.t. $(\mathcal{W}_t, P)$ for any $\phi \in C_0^\infty(\mathbf{R}^d)$. This implies that

$$E \left[\int_0^\infty e^{-t\lambda} [I - U_\lambda] \phi(X_{s+t}) dt \mid \mathcal{W}_s \right] = G_\lambda \phi(X_s),$$

for $(\lambda + p(x, D_x)) G_\lambda \phi(x) = [I - U_\lambda]\phi(x)$. Then the equality

$$E \left[\int_0^\infty e^{-t\lambda} f(X_{s+t}) dt \mid \mathcal{W}_s \right] = R_\lambda f(X_s)$$

is satisfied for any function $f \in [I - U_\lambda]C_0^\infty(\mathbf{R}^d)$. It holds that, for sufficiently large p, $\|R_\lambda f\|_\infty \leq c_\lambda\ \|f\|_{L^p}$ $(\lambda \geq \lambda_0)$. Since the space $[I - U_\lambda]C_0^\infty(\mathbf{R}^d)$ is dense in L^p, the property in the above Lemma holds good if the "L^p−estimate "

$$\left| E \left[\int_0^\infty e^{-t\lambda} f(X_{s+t}) dt \mid \mathcal{W}_s \right] \right| \leq c_\lambda\ \|f\|_{L^p}$$

holds good.

To prove the "L^p-estimate", we define a sequence of stable-like processes

$$(\widetilde{W}, (\widetilde{\mathcal{W}}_t), \widetilde{P}; \widetilde{X}_t^n)$$

with perturbations, whose laws approximate the law of the solution (X_t, P) to the martingale problem for L. Let J_X denote the counting measure of jumps of X :

$$J_X(dt, dy) = \#\{\tau \mid \tau \in dt, 0 \neq X_\tau - X_{\tau-} \in dy\}.$$

Set $\gamma_t = \alpha(X_t)$ and

$$M(x, \gamma, dy) = m(x, \gamma, y/|y|)|y|^{-d-\gamma}\ dy.$$

We see that the measure $J_X(dt,dy) - (M(X_t,\gamma_t,dy) + N(X_t,dy))dt$ is a martingale random measure w.r.t. $(\mathcal{W}_t, P)$. Let $a(x)$ be the $\mathbf{R}^d \otimes \mathbf{R}^d$−valued continuous function such that

$$\psi(x,2,D_x)f(x) = (1/2)\ tr[aa^*(x)\ (\partial\partial^* f(x))],$$

that is,

$$a(x)a^*(x) = \int_{|\omega|=1} \omega\omega^*[\partial_\gamma m(x,\gamma,\omega)]_{\gamma=2}\ \sigma(d\omega).$$

Set

$$b_0(x) := \psi(x,1,D_x)x = -\int_{|\omega|=1} \omega[\partial_\gamma m(x,\gamma,\omega)]_{\gamma=1}\ \sigma(d\omega).$$

Then there exists a continuous martingale $B_X(t)$ such that

$$\begin{aligned} dX_t \ &= \ a(X_t)\ dB_X(t) \ + \ (I_{(\gamma_t=1)}b_0(X_t) + I_{(\gamma_t>1+\varepsilon)}b(X_t))\ dt \\ &\quad + \int y\ [J_X(dt,dy) - I_{(\gamma_t>1)}M(X_t,\gamma_t,dy)dt - I_{(\gamma_t>1+\varepsilon,\ |y|\le 1)}N(X_t,dy)dt, \\ \langle dB_X^i(t), dB_X^j(t)\rangle \ &= \ \delta_{ij}\ I_{(\gamma_t=2)}\ dt. \end{aligned}$$

Let $Z = (Z_t)$ be a Cauchy process which is independent of $X = (X_t)$, and let J_Z denote the counting measure of jumps of Z. Set

$$\pi(n,t) = [nt]/n, \quad \omega_y = y/|y|, \quad \Theta_n(y) = y\ I_{(|y|\le 1/n)}, \quad \Theta_n^c(y) = y\ I_{(|y|>1/n)}.$$

Define processes $(\widetilde{X}_t^n)$ by the formula

$$\begin{aligned} d\widetilde{X}_t^n \ &= a(X_{\pi(n,t)})\ dB_X(t) \ + \ (I_{(\gamma_t=1)}b_0(X_{\pi(n,t)}) + I_{(\gamma_t>1+\varepsilon)}b(X_t))\ dt \\ &\quad + \int \Theta_n^c\left(\left[\frac{m(X_{\pi(n,t)},\gamma_t,\omega_y)}{m(X_t,\gamma_t,\omega_y)}\right]^{1/\gamma_t} y\right) \\ &\qquad \times [J_X(dt,dy) - I_{(\gamma_t>1)}M(X_t,\gamma_t,dy)dt - I_{(\gamma_t>1+\varepsilon,\ |y|\le 1)}N(X_t,dy)dt\ ] \\ &\quad + \int \Theta_n(\ [m(X_{\pi(n,t)},\gamma_t,\omega_y)|y|\ ]^{1/\gamma_t}\ \omega_y\ )\ [J_Z(dt,dy) - I_{(|y|\le 1)}|y|^{-d-1}dydt]. \end{aligned}$$

Since $m(x,\gamma,\omega),\ a(x),\ b(x)$ are continuous in x, it is a routine work to show that $\widetilde{X}_t^n \longrightarrow X_t$ in probability. We observe that, for $g(x) \in C_0^\infty(\mathbf{R}^d)$, there exists a martingale $(\widetilde{M}_t^n[g])$ such that

$$\begin{aligned} d\ g(\widetilde{X}_t^n) \ &= d\widetilde{M}_t^n[g] + (\psi(X_{\pi(n,t)},\gamma_t,D_x)g)(\widetilde{X}_t^n)\ dt \\ &\quad + \int [g(\widetilde{X}_t^n + \Theta_n^c) - g(\widetilde{X}_t^n) - I_{(\gamma_t>1+\varepsilon,|y|\le 1)}\Theta_n^c \cdot \partial_x g(\widetilde{X}_t^n)]N(X_t,dy)\ dt \\ &\quad + I_{(\gamma_t>1+\varepsilon)}b(X_t)\cdot\partial_x g(\widetilde{X}_t^n)\ dt, \end{aligned}$$

where $\Theta_n^c = \Theta_n^c([m(X_{\pi(n,t)},\gamma_t,\omega_y)/m(X_t,\gamma_t,\omega_y)]^{1/\gamma_t}y)$.

We have the following estimate (see [6], Lemma 1.1):

$$\begin{aligned} &\sup_{\gamma,|\xi|=1} \sum_{|\beta|\le d+1} \|\partial_\xi^\beta(\psi(\cdot,\gamma,\xi) - \psi(x_0,\gamma,\xi))\|_\infty \\ &\le\ c.\ \sup_{\gamma,|\omega|=1}\ [\ \sum_{|\beta|\le d} \|\partial_\omega^\beta(m(\cdot,\gamma,\omega) - m(x_0,\gamma,\omega))\|_\infty + \|\partial_\gamma(m(\cdot,\gamma,\omega) - m(x_0,\gamma,\omega))\|_\infty\ ]. \end{aligned}$$

Under the assumption that the value of $\|\alpha(\cdot) - \alpha(x_0)\|_\infty$ and the right-hand side of the above inequality are sufficiently small, from estimates for the operator $G_\lambda : L^p \longrightarrow C(\mathbf{R}^d) \bigcap L^p$, it can be proved that each process $(\widetilde{X}^n_t, \widetilde{P})$ admits the L^p-estimate

$$\left| \widetilde{E} \left[\int_0^\infty e^{-t\lambda} f(\widetilde{X}^n_{s+t}) dt \mid \widetilde{\mathcal{W}}_s \right] \right| \leq c_{n,\lambda} \, \|f\|_{L^p},$$

and that $c_\lambda := \sup_n c_{n,\lambda} < \infty$ (see Komatsu [7], Lemma 4.5). Then we obtain the L^p-estimate for the resolvent operators associated with solutions to the martingale problem for L, which implies the uniqueness of solutions to the martingale problem. The existence of solutions to the martingale problem for L can be proved under the same assumption (see [7], Theorem 3.1).

In the general case, to prove the existence and uniqueness of solutions to the martingale problem, we make use of " *the localization methods* ". Let $\rho_r(x) = \rho(|x - x_0|^2/r^2)$ and define $\alpha_r, \ \psi_r, L_r$ by

$$\begin{aligned} \alpha_r(x) &= \alpha(x_0) + \rho_r(x)(\alpha(x) - \alpha(x_0)), \\ \psi_r(x, \gamma, \xi) &= \psi(x_0, \gamma, \xi) + \rho_r(x)(\psi(x, \gamma, \xi) - \psi(x_0, \gamma, \xi)), \\ L_r &= \partial_t + \psi_r(x, \alpha_r(x), D_x) + \rho_r(x)\varphi(x, D_x). \end{aligned}$$

We see that, for sufficiently small $r > 0$, the existence and uniqueness of solutions to the martingale problem for L_r holds good. Set

$$T_r = \inf\{t \geq 0 \mid |X_t - x_0| > r\}.$$

Any local solution on $[0, T_r]$ to the martingale problem for L can be extended to a solution to the martingale problem for L_r. From the uniqueness of solutions to the martingale problem for L_r, we see that the local solution on $[0, T_r]$ to the martingale problem for L is uniquely determined. Repeating such localization methods, we see that the solution to the martingale problem for L exists and is uniquely determined.

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