ON ASYMPTOTIC BEHAVIOUR OF PROBABILITIES OF SMALL DEVIATIONS FOR COMPOUND COX PROCESSES

We derive logarithmic asymptotics for probabilities of small deviations for compound Cox processes in the space of trajectories. We find conditions under which these asymptotics are the same as those for sums of independent identically distributed random variables and homogeneous processes with independent increments. We show that if these conditions do not hold, the asymptotics of small deviations for compound Cox processes are quite different.
realization \( \lambda(t) \) of the measure \( \Lambda(t) \) the process \( N(t) \) is a non-homogeneous Poisson process with the intensity measure \( \lambda(t) \).

Now we define the compound Cox process in the same way as we introduce the processes \( \eta(t) \) and \( \zeta(t) \) above.

Let \( N(t) \) be a Cox process, independent with the sequence \( \{X_k\} \). The stochastic process \( S(t) = S_{N(t)} \) is called a compound Cox process.

Compound Cox processes play an important role in actuarial and financial mathematics. They describe, for example, the processes of total claims of an insurance company in a collective risk model (cf. [7]).

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Note that under additional conditions (cf., e.g. [7]), the Cox processes are renewal processes. Nevertheless, in the sequel, we assume that \( \Lambda(\cdot) \) is such that our Cox process will be the renewal process only if it is a Poisson process. So, we do not consider renewal processes here.

The logarithmic asymptotics of small deviations of the compound Cox processes has been investigated by Frolov [8]. In there, we have described the behaviour of

\[
P_T = P\left( \sup_{0 \leq t \leq T} \left| S(t) - c\Lambda(t) \right| \leq x_T \right),
\]

where \( \{x_T\} \) is a positive real function such that \( x_T \to \infty \) and \( x_T^2 = o(f(T)) \) as \( T \to \infty \), \( f(T) \) is a function, depending on properties of the measure \( \Lambda(t) \), \( c = EX \) for \( EX < \infty \) and \( c = 0 \) otherwise.

Note that if \( EX < \infty \), then \( S(t) \) is centered by a random function, generally speaking. Nevertheless, in the case of the homogeneous Poisson process, \( \Lambda(t) = \lambda t \) and our centering coincides with \( ES(t) \). The same holds true when \( N(t) \) is a non-homogeneous Poisson process. Moreover, centering functions of this type are used in appropriate models of the risk process. It turns out that fluctuations of risk in such models have to be compensate by insurance premiums of random amounts since, otherwise, the ruin probability of insurance company may be separated from zero and may be non-decreasing with increasing of initial capital of insurance company (cf. [7]). Therefore we consider this centering.

Here, we present generalizations of the results in Frolov [8]. \( P_T \) is the probability that trajectories of the compound Cox process, centered at \( c\Lambda(t) \) and normed by \( x_T \), lie in a strip of width 2 around zero. We consider below more general sets instead of strips and derive logarithmic asymptotics for such probabilities. We find conditions under which these asymptotics are the same as those for sums of independent identically distributed random variables and homogeneous processes with independent increments. We show that the asymptotics of small deviations for compound Cox processes are quite different, if these conditions do not hold.

1. Results

Let \( X, X_1, X_2, \ldots \) be a sequence of independent, identically distributed random variables. Put \( S_n = X_1 + X_2 + \cdots + X_n \) for \( n \geq 1 \), \( S_0 = 0 \).

Let \( \nu(t) \) be a standard Poisson process, independent with the sequence \( \{X_k\} \).

Denote \( \xi(t) = S_{\nu(t)} \). Note that \( \xi(t) \) is a compound Poisson process and, therefore, it is a homogeneous process with independent increments.

Let \( \Lambda(t), t > 0, \) be a random measure, i.e. a.s. (almost surely) \( \Lambda(0) = 0 \), \( \Lambda(t) < \infty \) for all \( t > 0 \) and \( \Lambda(t) \) has non-decreasing trajectories. Assume that the trajectories of \( \Lambda(t) \) are a.s. continuous and \( \Lambda(\infty) = \infty \) a.s. Suppose that \( \Lambda(t) \) is independent with the process \( \nu(t) \) and the sequence \( \{X_k\} \).

Define the Cox process \( N(t) \) and the compound Cox process \( S(t) \) by the relations \( N(t) = \nu(\Lambda(t)) \) and \( S(t) = S_{N(t)} \).

Let \( x_T \) be a real function with \( x_T \to \infty \) as \( T \to \infty \).
Let \( g_1(t), g_2(t), \ t \in [0,1], \) be continuous functions such that \( g_1(0) < 0 < g_2(0), \) \( g_1(t) \) is non-increasing and \( g_2(t) \) is non-decreasing. Put

\[
P_T = P \left( g_1 \left( \frac{\Lambda(t)}{\Lambda(T)} \right) x_T \leq S(t) - c\Lambda(t) \leq g_2 \left( \frac{\Lambda(t)}{\Lambda(T)} \right) x_T \text{ for all } t \in [0,T] \right).
\]

The probability \( P_T \) is well defined since trajectories of \( S(t) \) are jump functions.

In the sequel, we will describe the asymptotic behaviour of probabilities of small deviations \( P_T. \)

In general case, \( g_i(\Lambda(t)/\Lambda(T)) \) in the definition of \( P_T \) are random. We mentioned above that the random centering of \( S(t) \) is well motivated. So, random normings may also be used. Nevertheless, there are important cases in which \( g_i(\Lambda(t)/\Lambda(T)) \) are non-random functions.

The first case is that \( g_1(t) \equiv -1 \) and \( g_2(t) \equiv 1. \) In this case the results have been obtained in Frolov [8].

If \( N(t) \) is a Poisson process, then \( \Lambda(t) \) is a continuous increasing function. This is the second case.

The third case arises when \( \Lambda(t) = f(t), \) where \( f(t) \) is a continuous function. Then \( g_i(\Lambda(t)/\Lambda(T)) = g_i(f(t)/f(T)). \) If, in addition, \( f(t) = t^\beta, \beta > 0, \) we have

\[
P_T = P \left( g_3 \left( \frac{t}{T} \right) x_T \leq S(t) - c\Lambda(t) \leq g_4 \left( \frac{t}{T} \right) x_T \text{ for all } t \in [0,T] \right)
\]

where \( g_i(t+2) = g_i(t^\beta), \ i = 1,2. \) Note that in this last case, one can consider more general sets in the definition of \( P_T. \) To this end, consider a family of processes \( \{s_T(t) = S(T)/x_T, 0 \leq t \leq 1, T \in (0, \infty) \}. \) The trajectories \( s_T(\cdot) \) belong to the Skorohod space \( D[0,1] \) and we can introduce \( P(s_T(\cdot) \in G), \) where \( G \in \mathcal{G} \) and \( \mathcal{G} \) is a class of subsets of \( D[0,1]. \) One can define this class in the same way as in Mogul’skii [1] with the following additional assumption: on the first step of the definition on p. 757, functions \( L_1(t) \) and \( L_2(t)(L_1(t) > L_2(t)) \) have to be non-decreasing and non-increasing correspondingly. Of course, the class \( \mathcal{G} \) will be a subclass of the class in Mogul’skii [1], but it will be wide enough. It seems that the most interesting set is the set \( G_0 = \{g \in D[0,1] : g(0) = 0, g_3(t) < g(t) < g_4(t), \forall t \in [0,1] \}, \) where \( g_3(t), g_4(t), t \in [0,1], \) are continuous functions such that \( g_3(0) < 0 < g_4(0), g_3(t) \) is non-increasing and \( g_4(t) \) is non-decreasing. The asymptotics of \( P(s_T(\cdot) \in G_0) \) may be derived from our results below.

Our first result is the following theorem.

**Theorem 1.** Assume that there exist a positive, increasing, continuous function \( f(T), \) \( f(T) \to \infty \) as \( T \to \infty, \) and a non-negative random variables \( \Lambda \) such that the distributions of \( \Lambda(T)/f(T) \) converge weakly to the distribution \( \Lambda \) as \( T \to \infty. \) Denote \( a_T = \text{ess inf } \Lambda(T)/f(T), a = \text{ess inf } \Lambda. \) Assume that \( a_T \to a \) as \( T \to \infty \) and \( a > 0. \)

If \( EX < \infty, \) then put \( c = EX. \) Put \( c = 0 \) otherwise.

Suppose that the distribution of the random variable \( \xi_1 = \xi(1) - c \) belongs to a domain of attraction of a strictly stable law \( F_\alpha \) with the index \( \alpha \in (0,2], \) i.e. the distributions of \( (\xi(n) - cn)/B_n \) converge weakly to \( F_\alpha \) as \( n \to \infty, \) where \( \{B_n\} \) is a sequence of positive constants. Assume that \( F_\alpha \) is not concentrated on the half of line.

Then for every positive function \( x_T \) with \( x_T \to \infty \) and \( x_T \to o(B(f(T))) \) as \( T \to \infty, \) the following relation holds

\[
\log P_T = -CH_\alpha a \frac{f(T)}{x_T^\alpha} L(x_T)(1 + o(1)) \text{ as } T \to \infty,
\]
where \( L(x) = x^{\alpha-2}E \xi_1^2 I\{ |\xi_1| < x \} \) is a slowly varying at infinity function, \( C \) is an absolute positive constant, depending only on the distribution \( F_\alpha \), \( H_\alpha = \int_0^1 (g_2(t) - g_1(t))^{-\alpha} dt \).

If \( \alpha = 2 \), then \( C = \pi^2/8 \). Here and in the sequel, \( I\{ B \} \) denotes the indicator of the event \( B \), \( \lfloor x \rfloor \) denotes the integer part of \( x \).

Theorem 1 for \( g_1(t) \equiv -1 \) and \( g_2(t) \equiv 1 \) has been obtained in Frolov [8].

Conditions, necessary and sufficient for belonging of the distribution of \( \xi_1 \) to a domain of attraction of \( F_\alpha \), are well known. These conditions are usually stated in terms of asymptotic behaviours for tails or truncated moments (cf., for example, Feller [9], Chapter XVII, § 1). Nevertheless, to check the conditions of Theorem 1, it is more convenient to apply Theorem 2.6.5, p. 103 from Ibragimov and Linnik [10] where such conditions are given in terms of an asymptotic behaviour of a characteristic function at zero. Since 

\[
E e^{it\xi_1} = \exp \left\{ -itc + E e^{itX} - 1 \right\},
\]

one can easily check these conditions.

The simplest example of \( \Lambda(t) \) is \( \Lambda(t) = \Lambda f(t) \), where the random variable \( \Lambda \) and the function \( f(t) \) satisfy the conditions of Theorem 1. (Note that the random variable \( \Lambda \) may be degenerate and we will deal in this case with the non-homogeneous Poisson process \( N(t) \) and the corresponding process \( S(t) \).) We will arrive to another examples, if \( \Lambda(t) \) will be a stochastic process which satisfies the law of large numbers and has appropriate trajectories.

Theorem 1 yields that if \( a > 0 \), then the behaviour of \( \log P_T \) is the same as that for probabilities of small deviations for sums of independent identically distributed random variables and homogeneous processes with independent increments. Now we turn to the case \( a = 0 \). In this case the asymptotic of \( \log P_T \) may be quite different.

**Theorem 2.** Assume that the conditions of Theorem 1 hold and \( a = 0 \).

Then for every positive function \( x_T \) with \( x_T \to \infty \) and \( x_T = o(B(f(T))) \) as \( T \to \infty \), the following relation holds

\[
\log P_T \sim o \left( \frac{f(T)}{x_T^\alpha} L(x_T) \right) \quad \text{as} \quad T \to \infty,
\]

where \( L(x) \) is the function from Theorem 1.

In the case \( g_1(t) \equiv -1 \) and \( g_2(t) \equiv 1 \), Theorem 2 has been proved in Frolov [8].

Theorem 2 does not give the exact asymptotic of \( \log P_T \). In the next result we find this asymptotic which depends on the behaviour of the distribution functions of \( \Lambda(T)/f(T) \) and \( \Lambda \) at zero.

**Theorem 3.** Assume that the conditions of Theorem 1 hold and \( a_T = a = 0 \) for all \( T \). Put \( F_T(\lambda) = P(\Lambda(T) < \lambda f(T)) \) and \( F(\lambda) = P(\Lambda < \lambda) \).

For every positive function \( x_T \) with \( x_T \to \infty \) and \( x_T = o(B(f(T))) \) as \( T \to \infty \), define \( \varepsilon_T \) by the relation

\[
\varepsilon_T = \sup \left\{ \varepsilon > 0 : \frac{\varepsilon}{-\log F_T(\varepsilon)} \leq \frac{x_T^\alpha}{CH_\alpha f(T)L(x_T)} \right\},
\]

where the function \( L(x) \) and the constants \( C \) and \( H_\alpha \) are from Theorem 1. Assume that \( \varepsilon_T \) is equivalent to a continuous decreasing function and \( F_T(\varepsilon_T) \to 0 \) as \( T \to \infty \).

Suppose that for every \( \tau > 0 \), the following relation holds

\[
\log F_T(\tau \varepsilon_T) \sim \log F_T(\varepsilon_T) \quad \text{as} \quad T \to \infty.
\]

Then

\[
\log P_T \sim \log F_T(\varepsilon_T)(1 + o(1)) = -\varepsilon_T CH_\alpha \frac{f(T)}{x_T^\alpha} L(x_T)(1 + o(1)) \quad \text{as} \quad T \to \infty.
\]
Here $\varepsilon_T \to 0$ as $T \to \infty$.

Theorem 3 for $g_1(t) \equiv -1$ and $g_2(t) \equiv 1$ and continuous $F_T(\lambda)$ and $F(\lambda)$ has been obtained in Frolov [8].

Note that if for all $T$ the distribution functions $F_T(\lambda)$ and $F(\lambda)$ are continuous and positive in the non-degenerate interval $[0, \lambda]$, then in Theorem 3, $\varepsilon_T$ is a continuous decreasing function and $F_T(\varepsilon_T) \to 0$ as $T \to \infty$. Indeed, $F_T(\varepsilon_T) \leq \Delta + F(\varepsilon_T)$, where $\Delta = \sup_{0 \leq z \leq \lambda} |F_T(x) - F(x)| \to 0$ as $T \to \infty$ by the weak convergence of $F_T(\lambda)$ to $F(\lambda)$ and the continuity of the limit function.

We now show that the asymptotic of $\log P_T$ in (1) and (4) are quite different. To this goal, suppose that $F_T(x) \equiv F(x)$ for all $T$. If, for example, $F(x) = x^p$ for $x \in [0, 1]$, where $p > 0$, then $\log F_T(\varepsilon_T) \sim -p \log(\frac{f(T)}{x_T^p} L(x_T))$ as $T \to \infty$. If $F(x) = (-\log x)^{-p}$ for $x \in (0, e^{-1}]$, where $p > 0$, then $\log F_T(\varepsilon_T) \sim -p \log \log(\frac{f(T)}{x_T^p} L(x_T))$ as $T \to \infty$.

It turns out that the condition (3) cannot be omitted in Theorem 3. This follows from the next result.

**Theorem 4.** Assume that all the conditions of Theorem 3 hold except the condition (3). Assume that for all $\tau > 0$, the following relation holds $\log F_T(\tau \varepsilon_T) \sim \tau^p \log F_T(\varepsilon_T)$ as $T \to \infty$, where $p > 0$. Then

$$\log P_T = o \left( \varepsilon_T \frac{f(T)}{x_T^p} L(x_T) \right) \text{ as } T \to \infty.$$  

Theorem 4 for $g_1(t) \equiv -1$ and $g_2(t) \equiv 1$ and continuous $F_T(\lambda)$ and $F(\lambda)$ has been proved in Frolov [8].

2. Proofs

Put $\xi(t) = \xi(t) - ct$.

In what follows, we will use the following result.

**Lemma 1.** If the function $f(t)$ satisfies the conditions of Theorem 1, then the probability $P(g_1(t)x_T \leq \xi(\lambda f(T)t) \leq g_2(t)x_T \text{ for all } t \in [0, 1])$ is non-increasing in $\lambda$.

**Proof of Lemma 1.** Take $\lambda > 1$. Then

$$P(g_1(t)x_T \leq \xi(\lambda f(T)t) \leq g_2(t)x_T \text{ for all } t \in [0, 1])$$

$$= P \left( g_1 \left( \frac{u}{\lambda f(T)} \right) x_T \leq \xi(u) \leq g_2 \left( \frac{u}{\lambda f(T)} \right) x_T \text{ for all } u \in [0, \lambda f(T)] \right)$$

$$\leq P \left( g_1 \left( \frac{u}{\lambda f(T)} \right) x_T \leq \xi(u) \leq g_2 \left( \frac{u}{\lambda f(T)} \right) x_T \text{ for all } u \in [0, f(T)] \right)$$

$$\leq P \left( g_1 \left( \frac{u}{f(T)} \right) x_T \leq \xi(u) \leq g_2 \left( \frac{u}{f(T)} \right) x_T \text{ for all } u \in [0, f(T)] \right)$$

$$= P(g_1(t)x_T \leq \xi(f(T)t) \leq g_2(t)x_T \text{ for all } t \in [0, 1]).$$

In the last inequality we have used that $g_1(t)$ is non-increasing and $g_2(t)$ is non-decreasing.

We also need the following result on asymptotics of small deviations for compound Poisson process $\xi(t)$.

**Lemma 2.** Let $g(T)$ be an increasing, continuous function with $g(T) \to \infty$ as $T \to \infty$. 
If the conditions of Theorem 1 hold, then for every positive function \( x_T \) with \( x_T \to \infty \) and \( x_T = o(B_{\|f(T)\|}) \) as \( T \to \infty \), the following relation holds

\[
\log P \left( g_1(t)x_T \leq \tilde{\xi}(g(T)t) \leq g_2(t)x_T \quad \text{for all} \quad t \in [0, 1] \right) = -CH_\alpha \frac{g(T)}{x_T^{\alpha}} L(x_T)(1 + o(1))
\]

as \( T \to \infty \), where \( L(x) \) and \( C, H_\alpha \) are the function and the constants from Theorem 1.

Proof of Lemma 2. Take a sequence \( \{T_k\} \) such that \( T_k \not
\to \infty \) as \( k \to \infty \).

Since \( \tilde{\xi}(t) \) is a homogeneous process with independent increments, we have by Mogul’skii [1] and Lemma 1

\[
P \left( g_1(t)x_{T_k} \leq \tilde{\xi}(g(T_k)t) \leq g_2(t)x_{T_k} \quad \text{for all} \quad t \in [0, 1] \right) \leq P \left( g_1(t)x_{T_k} \leq \tilde{\xi}(g(T_k)t) \leq g_2(t)x_{T_k} \quad \text{for all} \quad t \in [0, 1] \right) = \exp \left\{ -CH_\alpha \frac{g(T_k)}{x_{T_k}^{\alpha}} L(x_{T_k})(1 + o(1)) \right\} \quad \text{as} \quad k \to \infty.
\]

The lower bound for the probability in (6) may be derived in the same way. Taking into account that the sequence \( \{T_k\} \) may be chosen arbitrarily, we get (6).

Proof of Theorem 1. Put \( F_T(\lambda) = P(\Lambda(T) < \lambda f(T)) \).

Taking into account Lemma 1 and the independence of \( \tilde{\xi}(t) \) and \( \Lambda(t) \), we have

\[
P_T = P \left( g_1 \left( \frac{\Lambda(t)}{\Lambda(T)} \right) x_T \leq \tilde{\xi}(\Lambda(t)) \leq g_2 \left( \frac{\Lambda(t)}{\Lambda(T)} \right) x_T \quad \text{for all} \quad t \in [0, T] \right)
\]

\[
= P(g_1(t)x_T \leq \tilde{\xi}(\Lambda(T)t) \leq g_2(t)x_T \quad \text{for all} \quad t \in [0, 1])
\]

\[
= \int_0^\infty P(g_1(t)x_T \leq \tilde{\xi}(\lambda f(T)t) \leq g_2(t)x_T \quad \text{for all} \quad t \in [0, 1])dF_T(\lambda)
\]

\[
\leq P(g_1(t)x_T \leq \tilde{\xi}(\lambda f(T)t) \leq g_2(t)x_T \quad \text{for all} \quad t \in [0, 1])
\]

\[
\leq P(g_1(t)x_T \leq \tilde{\xi}(a_T f(T)t) \leq g_2(t)x_T \quad \text{for all} \quad t \in [0, 1]).
\]

Take \( \varepsilon \in (0, a) \). Then \( a_T \geq a - \varepsilon \) for all sufficiently large \( T \). By Lemma 1 it follows that

\[
P_T \leq P(g_1(t)x_T \leq \tilde{\xi}((a - \varepsilon)f(T)t) \leq g_2(t)x_T \quad \text{for all} \quad t \in [0, 1]).
\]

for all sufficiently large \( T \).

The norming constants \( B_\alpha \) may be chosen such that \( B_\alpha = n^{1/\alpha}L_1(n) \), where \( L_1(x) \) is a slowly varying at infinity function (cf., for example, [10], p. 48). It follows that the condition \( x_T = o(B_{\|f(T)\|}) \) as \( T \to \infty \) is equivalent to the condition \( x_T = o(B_{b\|f(T)\|}) \) as \( T \to \infty \), where \( b \) is an arbitrary fixed positive constant. By Lemma 2

\[
\log P \left( g_1(t)x_T \leq \tilde{\xi}((a - \varepsilon)f(T)t) \leq g_2(t)x_T \quad \text{for all} \quad t \in [0, 1] \right)
\]

\[
= -CH_\alpha (a - \varepsilon) \frac{f(T)}{x_T^{\alpha}} L(x_T)(1 + o(1))
\]

as \( T \to \infty \). The latter and (8) yield that

\[
\limsup_{T \to \infty} \frac{x_T^{\alpha}}{f(T)L(x_T)} \log P_T \leq -CH_\alpha (a - \varepsilon).
\]

Taking in the last inequality the limit as \( \varepsilon \to 0 \), we get

\[
\limsup_{T \to \infty} \frac{x_T^{\alpha}}{f(T)L(x_T)} \log P_T \leq -CH_\alpha a.
\]
Take \( \varepsilon > 0 \). Since \( a_T \to a \) as \( T \to \infty \), we get from (7) and Lemma 1 that

\[
\begin{align*}
Pt &\geq \int_{r_T}^{(1+\varepsilon)^2x_T} P(g_1(t)x_T \leq \xi(\lambda f(T)t) \leq g_2(t)x_T \text{ for all } t \in [0,1])dF_T(\lambda) \\
&\geq P(g_1(t)x_T \leq \xi((1+\varepsilon)^2a_T f(T)t) \leq g_2(t)x_T \text{ for all } t \in [0,1])F_T((1+\varepsilon)^2\alpha_T) \\
&\geq P(g_1(t)x_T \leq \xi((1+\varepsilon)^3a_T f(T)t) \leq g_2(t)x_T \text{ for all } t \in [0,1])F_T((1+\varepsilon)a)
\end{align*}
\]

for all sufficiently large \( T \). Choose \( \varepsilon \) such that \( (1+\varepsilon)a \) is a point of continuity for \( F(\lambda) = P(\Lambda < \lambda) \). By the weak convergence of \( F_T(\lambda) \) to \( F(\lambda) \) we have

\[
(10) \quad P_T \geq \frac{1}{2}P(g_1(t)x_T \leq \xi((1+\varepsilon)^3a_T f(T)t) \leq g_2(t)x_T \text{ for all } t \in [0,1])F((1+\varepsilon)a)
\]

for all sufficiently large \( T \). Since \( x_T = o(B_T((1+\varepsilon)^3a_T f(T)t)) \) as \( T \to \infty \), by Lemma 2

\[
\log P \left( g_1(t)x_T \leq \xi((1+\varepsilon)^3a_T f(T)t) \leq g_2(t)x_T \text{ for all } t \in [0,1] \right) = -C(1+\varepsilon)^3a_T f(T) L(x_T) (1+o(1))
\]

as \( T \to \infty \). It follows from (10) that

\[
(11) \quad \liminf_{T \to \infty} \frac{x_T^2}{f(T)L(x_T)} \log P_T \geq -CaH_a(1+\varepsilon)^3.
\]

Taking in the last inequality the limit as \( \varepsilon \to 0 \), we get

\[
\liminf_{T \to \infty} \frac{x_T^2}{f(T)L(x_T)} \log P_T \geq -CaH_a,
\]

which together with (9) yields (1).

**Proof of Theorem 2.** It is clear that \( \log P_T \leq 0 \) and we need only prove the lower bound.

Take \( \varepsilon > 0 \). Using (7) and Lemma 1, we have

\[
Pt \geq \int_{r_T}^{(1+\varepsilon)^2x_T} P(g_1(t)x_T \leq \xi(\lambda f(T)t) \leq g_2(t)x_T \text{ for all } t \in [0,1])dF_T(\lambda) \\
\geq P(g_1(t)x_T \leq \xi((1+\varepsilon)^2\varepsilon f(T)t) \leq g_2(t)x_T \text{ for all } t \in [0,1])F_T((1+\varepsilon)^2\varepsilon) - F_T(\varepsilon)
\]

for all sufficiently large \( T \). In the same way as in the proof of Theorem 1, the last implies (11) with \( \varepsilon \) instead of \( a \). Taking the limit as \( \varepsilon \to 0 \), we get (2).

**Proof of Theorem 3.** Put \( b_T = CH_a \frac{f(T)}{x_T^3} L(x_T) \). The condition \( x_T = o(B_T(f(x_T))) \) as \( T \to \infty \) and formulae for the norming constants \( B_n \), which may be chosen to satisfy

\[
(12) \quad \frac{nL(B_n)}{B_n^3} \to d \quad \text{as} \quad n \to \infty,
\]

(cf., for example, [9], Chapter XVII, 85), imply that \( b_T \to \infty \) as \( T \to \infty \).

We will first prove that \( \varepsilon_T \to 0 \) as \( T \to \infty \).

Suppose that there exists a sequence \( \{T_k\} \) such that \( T_k \not\to \infty \) as \( k \to \infty \) and \( \varepsilon_{T_k} > \varepsilon > 0 \) for all sufficiently large \( k \), where \( \varepsilon \) is a point of continuity of \( F(\lambda) \). Then \( -\log F_{T_k}(\varepsilon_{T_k}) \leq -\log F_{T_k}(\varepsilon) \leq -\log(F(\varepsilon)/2) < \infty \) for all sufficiently large \( k \) in view of the weak convergence of \( F_T(\lambda) \) to \( F(\lambda) \). Hence \( 1/b_{T_k} \geq -\varepsilon_{T_k}/\log F_{T_k}(\varepsilon_{T_k}) > -\varepsilon/\log(F(\varepsilon)/2) > 0 \) which contradicts to the relation \( b_T \to \infty \) as \( T \to \infty \).

By the definition of \( \varepsilon_T \), we get \(-3\varepsilon_T/\log(F_T(3\varepsilon_T)) > 1/b_T \). This and (3) imply that \(-2\varepsilon_T/\log(F_T(\varepsilon_T)) \geq 1/b_T \) for all sufficiently large \( T \). The latter and the relation
Proof of Theorem 4. As in the proof of Theorem 2, we need only prove the lower bound.

We have from (7) and Lemma 1

\[ P_T = \int_0^{\varepsilon_T} P(g_1(t)x_T \leq \xi(\lambda f(T)t) \leq g_2(t)x_T \text{ for all } t \in [0, 1])dF_T(\lambda) \]

\[ \quad + \int_{\varepsilon_T}^{\infty} P(g_1(t)x_T \leq \xi(\lambda f(T)t) \leq g_2(t)x_T \text{ for all } t \in [0, 1])dF_T(\lambda) \]

\[ \leq F_T(\varepsilon_T) + P(g_1(t)x_T \leq \xi(\varepsilon_T f(T)t) \leq g_2(t)x_T \text{ for all } t \in [0, 1]) \]

\[ \leq e^{-\varepsilon_T b_T} + P(g_1(t)x_T \leq \xi(\varepsilon_T f(T)t) \leq g_2(t)x_T \text{ for all } t \in [0, 1]). \]

Applying Lemma 2, we get the upper bound in (4).

We now turn to the lower bound. Take \( \tau > 0 \). By Lemmas 1 and 2

\[ P_T \geq \int_0^{\varepsilon_T} P(g_1(t)x_T \leq \xi(\lambda f(T)t) \leq g_2(t)x_T \text{ for all } t \in [0, 1])dF_T(\lambda) \]

\[ \geq P(g_1(t)x_T \leq \xi(\tau \varepsilon_T f(T)t) \leq g_2(t)x_T \text{ for all } t \in [0, 1])F_T(\tau \varepsilon_T) \]

\[ = e^{-\tau \varepsilon_T b_T(1+o(1))} F_T(\tau \varepsilon_T) \]

as \( T \to \infty \). By the definition of \( \varepsilon_T \) we get \( \log F_T((1 + \tau)\varepsilon_T) > -(1 + \tau)\varepsilon_T b_T \). It follows from the last inequality and (3) that

\[ P_T \geq e^{-\tau \varepsilon_T b_T(1+o(1))} \log F_T((1 + \tau)\varepsilon_T(1+o(1))) \geq e^{-(1+2\tau)\varepsilon_T b_T(1+o(1))} \]

as \( T \to \infty \). Since \( \tau \) may be chosen arbitrarily small, we get the lower bound in (4).

Proof of Theorem 4. As in the proof of Theorem 2, we need only prove the lower bound.

In the same way as in proof of (13), we get

\[ P_T \geq e^{-(1+2\tau)\varepsilon_T b_T(1+o(1))} \]

as \( T \to \infty \) which yields (5).


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