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A UNIQUENESS THEOREM FOR THE MARTINGALE PROBLEM DESCRIBING A DIFFUSION IN MEDIA WITH MEMBRANES

We formulate a martingale problem that describes a diffusion process in a multidimensional Euclidean space with a membrane located on a given smooth surface and having the properties of skewing and delaying. The theorem on the existence of no more than one solution to the problem is proved.

INTRODUCTION

Let S be a given closed bounded surface in \mathbb{R}^d that divides the space \mathbb{R}^d into two open parts: the interior domain D^i and the exterior one D^e , D is the union of them. The surface S is assumed to be smooth enough (see Section 1 for the precise assumptions) so that there is a well-defined normal at any point of S . By $\nu(x)$ for $x \in S$, we denote the unit vector of the outward normal to S at the point x . Let $A(x)$, $x \in S$, be a given real-valued continuous function and, for each $y \in \mathbb{R}^d$, let $b(y)$ be a symmetric positive definite linear operator in \mathbb{R}^d . The function $(b(y))_{y \in \mathbb{R}^d}$ is supposed to be bounded and Hölder continuous. For $x \in S$, the vector $N(x) = b(x)\nu(x)$ is called the co-normal vector to S at the point x . Consider the stochastic differential equation in \mathbb{R}^d

$$(1) \quad dx(t) = A(x(t))N(x(t))\mathbb{I}_S(x(t))dt + b(x(t))^{1/2}\mathbb{I}_D(x(t))dw(t),$$

where $(w(t))_{t \geq 0}$ is a standard Wiener process in \mathbb{R}^d , \mathbb{I}_Γ is the indicator function of a set $\Gamma \subset \mathbb{R}^d$. As was shown in [1], this equation has infinitely many solutions. Consequently, if a solution to (1) is treated as that to the corresponding martingale problem, the latter turns out not to be well-posed.

Each solution constructed in [1] is determined by a representation of the function $A(x)$, $x \in S$, in the form $A(x) = \frac{q(x)}{r(x)}$, where $q(\cdot)$ and $r(\cdot)$ are continuous functions on S taking their values in $[-1, 1]$ and $(0, +\infty)$, respectively. Thus, the formulation of the well-posed martingale problem must involve these functions.

A solution to (1) was constructed in [1] as a continuous Markov process $(x(t))_{t \geq 0}$ in \mathbb{R}^d obtained from a d -dimensional diffusion process with its diffusion operator $b(\cdot)$ and a zero drift vector by two transformations. The first transformation is skewing the diffusion process on S . The skew is determined by the function $q(\cdot)$. As a result, one gets a continuous Markov process $(x_0(t))_{t \geq 0}$ in \mathbb{R}^d such that its trajectories satisfy the stochastic differential equation (see [2], Ch. 3)

$$(2) \quad dx_0(t) = q(x_0(t))\delta_S(x_0(t))N(x_0(t))dt + b(x_0(t))^{1/2}dw(t),$$

where $(\delta_S(x))_{x \in \mathbb{R}^d}$ is a generalized function on \mathbb{R}^d that acts on a test function $(\varphi(x))_{x \in \mathbb{R}^d}$ according to the following rule:

$$\langle \delta_S, \varphi \rangle = \int_S \varphi(x) d\sigma$$

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(the integral in this equality is a surface integral).

To do the second transformation determined by a given function $r(\cdot) : S \rightarrow (0, \infty)$, one should put, for $t \geq 0$,

$$\zeta_t = \inf \left\{ s : s + \int_0^s r(x_0(\tau)) \delta_S(x_0(\tau)) d\tau \geq t \right\}$$

and define

$$x(t) = x_0(\zeta_t), \quad t \geq 0.$$

Here, the functional

$$\eta_t = \int_0^t r(x_0(\tau)) \delta_S(x_0(\tau)) d\tau, \quad t \geq 0,$$

of the process $(x_0(t))_{t \geq 0}$ is well defined as an additive homogeneous continuous functional (see [2], Ch. 3). As is known (see [3], Theorem 10.11), the process $(x(t))_{t \geq 0}$ is a continuous Markov process in \mathbb{R}^d as a result of the random change of time for the process $(x_0(t))_{t \geq 0}$.

The following observation gives us a suggestion how to formulate correctly the martingale problem for the process $(x(t))_{t \geq 0}$ corresponding to a given pair of functions $q(\cdot)$ and $r(\cdot)$. Namely, fix an orthonormal basis in \mathbb{R}^d and denote, by x_j for $j = 1, 2, \dots, d$ the coordinates of a vector $x \in \mathbb{R}^d$ and by $b_{jk}(x)$ for $j, k = 1, 2, \dots, d$, the elements of the matrix of the operator $b(x)$ in that basis. For a given continuous bounded function φ on \mathbb{R}^d with real values, we put $u(t, x, \varphi) = \mathbb{E}_x \varphi(x(t))$, $t \geq 0$ and $x \in \mathbb{R}^d$. Then this function is continuous in the arguments $t \geq 0$ and $x \in \mathbb{R}^d$ and turns out to satisfy the following conditions:

1) it satisfies the equation

$$(3) \quad \frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j=1}^d b_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}$$

in the domain $t > 0$, $x \in D$;

2) it satisfies the equation

$$r(x) \frac{\partial u}{\partial t} = \frac{1+q(x)}{2} \frac{\partial u(t, x+)}{\partial N(x)} - \frac{1-q(x)}{2} \frac{\partial u(t, x-)}{\partial N(x)}$$

for $t > 0$, $x \in S$;

3) the initial condition

$$u(0+, x) = \varphi(x)$$

holds for all $x \in \mathbb{R}^d$.

In Section 1, we give a correct form of the martingale problem desired. Our aim is to show that the solution to that problem is unique. We obtain the statement from the uniqueness theorem for the boundary process (see Section 2 for the precise definition) by the Strook–Varadhan method from [4]. The particular case of an identity diffusion matrix and S being a hyperplane was investigated in [5]. One can also find there some further discussion of the topic.

1. THE MARTINGALE PROBLEM

From now on, we assume that, for each $x \in \mathbb{R}^d$, $b(x) = (b_{ij}(x))_{i,j=1}^d$ is a symmetric $d \times d$ -matrix satisfying the following conditions we call the conditions J :

1) there are two positive constants C_1 and C_2 , $0 < C_1 \leq C_2$, such that

$$C_1 |\theta|^2 \leq (b(x)\theta, \theta) \leq C_2 |\theta|^2$$

is valid for all $\theta, x \in \mathbb{R}^d$.

2) for all $x, x' \in \mathbb{R}^d$, $i, j = 1, 2, \dots, d$,

$$(4) \quad |b_{ij}(x) - b_{ij}(x')| \leq L|x - x'|^\alpha,$$

where L and α are positive constants, $\alpha \leq 1$.

Suppose S belongs to the class $H^{2+\varkappa}$ for some $\varkappa \in (0, 1)$ (see [6], Ch. 4, § 4). By δ , we denote the minimal one of the numbers α from (4) and \varkappa .

Suppose a continuous function $q(\cdot) : S \rightarrow [-1, 1]$ and a continuous bounded function $r(\cdot) : S \rightarrow [0, +\infty)$ are fixed.

Ω stands for the space of all continuous \mathbb{R}^d -valued functions on $[0, +\infty)$, and \mathcal{M}_t denotes the σ -algebra generated by $x(u)$ for $0 \leq u \leq t$. If $t = \infty$, \mathcal{M}_t will be denoted by \mathcal{M} .

We say that a function f belongs to the class F if

- 1) f is continuous and bounded in (t, x) on $[0, +\infty) \times \mathbb{R}^d$;
- 2) f has a continuous and bounded derivative with respect to t on $[0, +\infty) \times \mathbb{R}^d$;
- 3) f has continuous and bounded derivatives with respect to x on $[0, +\infty) \times D$ up to the second order;

4) for all $t \in [0, +\infty)$ and $x \in S$, there exist the non-tangent limits $\frac{\partial f(t, x+)}{\partial N(x)}$ and $\frac{\partial f(t, x-)}{\partial N(x)}$ from the side D^e and D^i , respectively, and the function

$$Kf(t, x) = \frac{1+q(x)}{2} \frac{\partial f(t, x+)}{\partial N(x)} - \frac{1-q(x)}{2} \frac{\partial f(t, x-)}{\partial N(x)}$$

is continuous and bounded on $[0, +\infty) \times S$.

Definition 1. Given $x \in \mathbb{R}^d$, a probability measure \mathbb{P}_x on \mathcal{M} is a solution to the submartingale problem starting from x if

- 1) $\mathbb{P}_x\{x(0) = x\} = 1$;
- 2) the process

$$X_f(t) = f(t, x(t)) - \int_0^t \mathbb{1}_D(x(u)) \left(\frac{\partial f}{\partial u} + \frac{1}{2} \sum_{i,j=1}^d b_{ij}(x(u)) \frac{\partial^2 f}{\partial x_i \partial x_j} \right) (u, x(u)) du, \quad t \geq 0,$$

is a \mathbb{P}_x -submartingale whenever f belongs to F and satisfies the inequality

$$r(x) \frac{\partial f(t, x)}{\partial t} + Kf(t, x) \geq 0 \text{ for } t \geq 0 \text{ and } x \in S.$$

Remark 1. One can verify that the transition probability of the process $(x(t))_{t \geq 0}$ described in the Introduction is a solution to the submartingale problem ([1]).

Define the function ϕ on \mathbb{R}^d by the equality $\phi(x) = d(x, S) := \inf\{d(x, y) : y \in S\}$, where $d(\cdot, \cdot)$ is the Euclidean metric on \mathbb{R}^d . Then

- 1) $S = \{x \in \mathbb{R}^d : \phi(x) = 0\}$, $D = \{x \in \mathbb{R}^d : \phi(x) > 0\}$,
- 2) $K\phi(x) \equiv 1$ on S .

Remark 2. The function ϕ does not belong to the class F because of its unboundedness. To overcome this, we choose, for each $m \geq 1$, a non-increasing infinitely differentiable function η_m defined on $[0, \infty)$ and having a compact support such that $0 \leq \eta_m \leq 1$, $\eta_m \equiv 1$ on $[0, m]$, $\eta_m \equiv 0$ off $[0, m+1]$ and the derivatives of η_m up to the second order are uniformly bounded. Set $\phi_m(x) = \eta_m(d(x, S)) \cdot \phi(x)$, $x \in \mathbb{R}^d$. Then ϕ_m belongs to F . Hence, X_{ϕ_m} is a \mathbb{P}_x -submartingale. Clearly, $\phi_m(x) \rightarrow \phi(x)$ monotonically as $m \rightarrow \infty$ and $\sum_{i,j=1}^d b_{ij}(x) \frac{\partial^2 \phi_m(x)}{\partial x_i \partial x_j}$ tends to 0 boundedly. So $X_\phi(t) = \phi(x(t))$ is a \mathbb{P}_x -submartingale.

The following proposition gives a reformulation of the submartingale problem into a martingale one.

Proposition 1. *Given $x \in \mathbb{R}^d$, the probability measure \mathbb{P}_x on \mathcal{M} solves the submartingale problem starting from x iff $\mathbb{P}_x\{x(0) = x\} = 1$ and there exists a continuous non-decreasing (\mathcal{M}_t) -adapted process $\gamma(t)$, $t \geq 0$, such that*

- 1) $\gamma(0) = 0$, $\mathbb{E}\gamma(t) < +\infty$ for all $t \geq 0$;
- 2) $\gamma(t) = \int_0^t \mathbb{I}_S(x(u))d\gamma(u)$, $t \geq 0$;
- 3) the process

$$f(t, x(t)) - \int_0^t \mathbb{I}_D(x(u)) \left(\frac{\partial f}{\partial u} + \frac{1}{2} \sum_{i,j=1}^d b_{ij}(x(u)) \frac{\partial^2 f}{\partial x_i \partial x_j} \right) (u, x(u)) du - \\ - \int_0^t \left(r \frac{\partial f}{\partial u} + Kf \right) (u, x(u)) d\gamma(u), t \geq 0,$$

is a \mathbb{P}_x -martingale for any f belonging to F .

If \mathbb{P}_x is such a solution, then $\gamma(t)$ is uniquely determined, up to \mathbb{P}_x -equivalence, by the condition that

$$\phi(x(t)) - \gamma(t), t \geq 0,$$

is a \mathbb{P}_x -martingale.

Proof. The existence of a solution to this problem was established in [7]. The proof of the last statement is similar to that of Theorem 2.5 in [4].

Corollary 1. *For each $x \in \mathbb{R}^d$, $t \geq 0$, the equality*

$$\int_0^t \mathbb{I}_S(x(u)) du = \int_0^t r(x(u)) d\gamma(u)$$

holds \mathbb{P}_x -almost surely.

Corollary 2. *If $x \in S$, then $\mathbb{P}_x\{\gamma(t) > 0, t > 0\} = 1$.*

These assertions can be verified like Corollaries 1,2 in [5].

2. A UNIQUENESS THEOREM FOR A BOUNDARY PROCESS

Let \mathbb{P}_x be a solution to the submartingale problem starting from $x \in S$. Then there exists a process $\gamma(t)$, $t \geq 0$, that has the properties stated in Proposition 1. For $\theta \geq 0$, we put $\tau(\theta) = \sup\{t \geq 0 : \gamma(t) \leq \theta\}$. Define $T(\omega) = \lim_{t \rightarrow +\infty} \gamma(t)$. Assume that $T(\omega) = +\infty$ a.s. Then the process $y(\theta) = x(\tau(\theta))$ is defined for all $0 \leq \theta < \infty$. It is not hard to see that the process $\tau(\theta)$ and, consequently, the process $y(\theta)$ are right-continuous processes having no discontinuities of the second kind, and the latter takes on its values on S . Since the starting point is on S , we have $\gamma(t) > 0$ for $t > 0$ almost surely, i.e., $\tau(0) = 0$ and $y(0) = x$. Following Strook and Varadhan [4], we define a $(d+1)$ -dimensional process $(\tau(\theta), y(\theta))$, $\theta \geq 0$, and call it the boundary process starting from x . If $T(\omega) < \infty$ with positive probability, we put $(\tau(\theta), y(\theta)) = \infty$ for $\theta \geq T(\omega)$.

Further on we denote, by $C_0^{1,2}([0, +\infty) \times S)$, the class of functions on $[0, +\infty) \times S$ that have compact supports with respect to t and together with their first t -derivative and two x -derivatives are continuous and bounded, $C_0^\infty([0, +\infty) \times S)$ stands for the class of infinitely differentiable functions on $[0, +\infty) \times S$ having compact supports with respect to t .

Proposition 2. For each $h \in C_0^{1,2}([0, +\infty) \times S)$, there exists a function Hh such that

- (i) it belongs to the class F ;
- (ii) it is a solution to the equation

$$(5) \quad \frac{\partial U}{\partial t} + \frac{1}{2} \sum_{i,j=1}^d b_{ij}(x) \frac{\partial^2 U}{\partial x^i \partial x^j} = 0$$

on both $[0, +\infty) \times D^i$ and $[0, +\infty) \times D^e$;

- (iii) the relations

$$(6) \quad Hh(t, x+) = h(t, x),$$

$$(7) \quad Hh(t, x-) = h(t, x),$$

hold true for all $t \geq 0$ and $x \in S$.

Proof. Assume that Θ is a domain in \mathbb{R}^d . Set $\Theta_T = (0, T) \times \Theta$ and denote, by $\overline{\Theta}_T$, its closure. Let $H^{\delta/2+1, \delta+2}(\overline{\Theta}_T)$ be a corresponding Hölder space (see [6]), $H_T^{\delta/2+1, \delta+2}(\overline{\Theta}_T)$ stands for the set of all functions from $H^{\delta/2+1, \delta+2}(\overline{\Theta}_T)$ which together with their first derivatives with respect to t are equal to zero at the point $t = T$. Notice that for all $T > 0$, $h \in H^{\delta/2+1, \delta+2}(\overline{S}_T)$. In addition, there exists $T_0 > 0$ such that $h = 0$ if $t \geq T_0$. Therefore, $h \in H_{T_0}^{\delta/2+1, \delta+2}(\overline{S}_{T_0})$. By analogy to Theorem 5.2 in [6], there is a uniquely defined function $h_{T_0}^i \in H_{T_0}^{\delta/2+1, \delta+2}(\overline{D}_{T_0}^i)$ which satisfy Eq. (5) in the domains $(0, T_0) \times D_{T_0}^i$ and boundary conditions (6). Remark that if $h(t, x) \equiv 0$ on $[A, B] \times S$, where A and B are some constants, $0 < A < B$, then the function equal to zero on $[A, B] \times \overline{D}^i$ is the unique solution to problem (5), (6) belonging to $H_B^{\delta/2+1, \delta+2}([A, B] \times \overline{D}_B^i)$. Similarly, there is a uniquely defined function $h_{T_0}^e \in H_{T_0}^{\delta/2+1, \delta+2}(\overline{D}_{T_0}^e)$ that is a solution to (5), (7) in $(0, T_0) \times D_{T_0}^e$. Now we define the function

$$Hh = \begin{cases} h_{T_0}^i & \text{on } \overline{D}_{T_0}^i, \\ h_{T_0}^e & \text{on } \overline{D}_{T_0}^e, \\ 0, & \text{otherwise.} \end{cases}$$

The function Hh has all the required properties, and this completes the proof.

Proposition 2 implies that, for each $h \in C_0^{1,2}([0, +\infty) \times S)$,

$$\begin{aligned} (\tilde{K}h)(t, x) &= r(x) \frac{\partial(Hh)(t, x)}{\partial t} + \\ &+ \left[\frac{1+q(x)}{2} \frac{\partial(Hh)(t, x+)}{\partial N(x)} - \frac{1-q(x)}{2} \frac{\partial(Hh)(t, x-)}{\partial N(x)} \right] \end{aligned}$$

is well defined as a continuous and bounded function on $[0, +\infty) \times S$.

Proposition 3. Suppose a probability measure \mathbb{P}_x solves the submartingale problem starting from $x \in S$. Then the relation $\mathbb{P}_x\{\tau(0), y(0) = (0, x)\} = 1$ holds, and, for any function $h \in C_0^{1,2}([0, +\infty) \times S)$, the process

$$h(\tau(\theta), y(\theta)) - \int_0^\theta (\tilde{K}h)(\tau(u), y(u)) du, \quad \theta \geq 0,$$

is a \mathbb{P}_x -martingale with respect to the filtration $(\mathcal{M}_{\tau(\theta)})_{\theta \geq 0}$.

Proof. The proof follows that of Theorem 4.1 in [4].

Denote, by $\mathcal{D}([0, +\infty), [0, +\infty) \times S)$, the class of $[0, +\infty) \times S$ -valued right-continuous functions on $[0, +\infty)$ with no discontinuities of the second kind.

Definition 2. The uniqueness theorem is valid for the boundary process if, for any given $x \in S$, there is only one probability measure \mathbb{Q}_x on the space $\mathcal{D}([0, +\infty), [0, +\infty) \times S)$ such that

- 1) $\mathbb{Q}_x\{\tau(0) = 0, y(0) = x\} = 1$;
- 2) $(\tau(\theta), y(\theta)) = \infty$ if $\theta > T(\omega)$;
- 3) for any function $h \in C_0^{1,2}([0, +\infty) \times S)$, the process

$$h(\tau(\theta), y(\theta)) - \int_0^\theta (\tilde{K}h)(\tau(u), y(u))du, \quad \theta \geq 0,$$

is a \mathbb{Q}_x -martingale relative to the natural σ -algebras $\tilde{\mathcal{M}}_\theta$, $\theta \geq 0$, in $\mathcal{D}([0, +\infty), [0, +\infty) \times S)$.

To prove the uniqueness theorem for the boundary process, we will make use of the following lemma.

Lemma. For each $\lambda > 0$, $\psi \in C_0^\infty([0, +\infty) \times S)$, the equation

$$(8) \quad \lambda f - \tilde{K}f = \psi$$

has a unique solution in the class of all continuous functions on $[0, \infty) \times S$ having compact supports with respect to t and such that there exists a function Hf on $[0, \infty) \times \mathbb{R}^d$ satisfying conditions (i)-(iii) of Proposition 2.

Proof. We first prove the Lemma in the case of r being identically equal to 0 on S .

Let $g_0(t, x, y)$, $t > 0$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$, be the fundamental solution to Eq. (3) (see [8], Ch. I). The process $(x_0(t))_{t \geq 0}$ solving Eq. (2) possesses a transition probability density. We denote it by $G_0(t, x, y)$, $t > 0$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$. As is proved in [1], for $t > 0$, $x \in \mathbb{R}^d$, and $y \in \mathbb{R}^d$, the representation

$$G_0(t, x, y) = g_0(t, x, y) + \int_0^t d\tau \int_S \tilde{V}(\tau, x, z) \frac{\partial g_0(t - \tau, z, y)}{\partial N(z)} q(z) d\sigma_z$$

takes place, where, for $t > 0$, $x \in \mathbb{R}^d$, and $y \in S$, $\tilde{V}(t, x, y)$ is the solution to the integral equation

$$\tilde{V}(t, x, y) = g_0(t, x, y) + \int_0^t d\tau \int_S \tilde{V}(\tau, x, z) \frac{\partial g_0(t - \tau, z, y)}{\partial N(z)} q(z) d\sigma_z.$$

In addition, the equality

$$(9) \quad \tilde{V}(t, x, y) = \frac{1}{2} [G_0(t, x, y+) + G_0(t, x, y-)]$$

is valid for $t > 0$, $x \in \mathbb{R}^d$, and $y \in S$, and the equality

$$(10) \quad \frac{1 + q(x)}{2} \frac{\partial G_0(t, x+, y)}{\partial N(x)} - \frac{1 - q(x)}{2} \frac{\partial G_0(t, x-, y)}{\partial N(x)} = 0$$

holds for $t > 0$, $x \in S$, and $y \in \mathbb{R}^d$.

For $\lambda \geq 0$, we define a function G_λ of the arguments $t > 0$, $x \in \mathbb{R}^d$, and $y \in \mathbb{R}^d$ by the relation

$$E_x(\varphi(x_0(t)) \exp\{-\lambda \eta_t\}) = \int_{\mathbb{R}^d} \varphi(y) G_\lambda(t, x, y) dy$$

that must be fulfilled for all $t > 0$, $x \in \mathbb{R}^d$, and φ being a bounded measurable function on \mathbb{R}^d . Then (see [1]) such a function exists, and it can be found as a solution to the pair of equations

$$(11) \quad G_\lambda(t, x, y) = G_0(t, x, y) - \lambda \int_0^t d\tau \int_S \tilde{V}(\tau, x, z) G_\lambda(t - \tau, z, y) r(z) d\sigma_z,$$

$$(12) \quad G_\lambda(t, x, y) = G_0(t, x, y) - \lambda \int_0^t d\tau \int_S G_\lambda(\tau, x, z) G_0(t - \tau, z, y) r(z) d\sigma_z$$

in the domain $t > 0$, $x \in \mathbb{R}^d$, and $y \in \mathbb{R}^d$. Moreover, there is no more than one solution to these equations satisfying the inequality

$$(13) \quad G_\lambda(t, x, y) \leq G_0(t, x, y).$$

Then, for all $t \geq 0$, $x \in \mathbb{R}^d$, we can define the function

$$V_\lambda(t, x) = \int_t^\infty d\tau \int_S G_\lambda^1(\tau - t, x, y) \psi(\tau, y) d\sigma_y,$$

where $G_\lambda^1(t, x, y)$ is the solution to the pair of equations (11), (12) satisfying inequality (13) for $r(x) \equiv 1$.

Notice that G_λ as a function of the third argument has a jump at the points of S . Namely, for $t > 0$, $x \in \mathbb{R}^d$, and $y \in S$, Eqs. (11) can be rewritten as

$$G_\lambda(t, x, y) = \tilde{V}(t, x, y) - \lambda \int_0^t d\tau \int_S \tilde{V}(\tau, x, z) G_\lambda(t - \tau, z, y) r(z) d\sigma_z.$$

From this, we can write the following relation for the function $V_\lambda(t, x)$:

$$(14) \quad V_\lambda(t, x) = \int_t^\infty d\tau \int_S \tilde{V}(\tau - t, x, y) \psi(\tau, y) d\sigma_y - \lambda \int_t^\infty d\tau \int_S \tilde{V}(\tau - t, x, y) V_\lambda(\tau, y) d\sigma_y.$$

The function $V_\lambda(t, x)$ has the following properties:

- 1) $V_\lambda(t, x)$ satisfies conditions (i), (ii) of Proposition 2;
- 2) the equality

$$(15) \quad \lambda V_\lambda(t, x) - \left[\frac{1 + q(x)}{2} \frac{\partial V_\lambda(t, x+)}{\partial N(x)} - \frac{1 - q(x)}{2} \frac{\partial V_\lambda(t, x-)}{\partial N(x)} \right] = \psi(t, x)$$

holds for $t \geq 0$ and $x \in S$.

Property 1) is easily justified. Applying the theorem on the jump of the co-normal derivative of a single-layer potential to (14) [6] or, more precisely, its version for the integrals over $[t, \infty)$ instead of the ones over $[0, t]$, we get the relations

$$\frac{\partial V_\lambda(t, x \pm)}{\partial N(x)} = \mp \psi(t, x) \pm \lambda V_\lambda(t, x) + \int_t^\infty d\tau \int_S \frac{\partial \tilde{V}(\tau - t, x, y)}{\partial N(x)} (\psi(\tau, y) - \lambda V_\lambda(\tau, y)) d\sigma_y$$

valid for $t > 0$ and $x \in S$.

Taking into account (9),(10), we arrive at formula (15).

Obviously, the restriction of the function $V_\lambda(t, x)$ on $[0, \infty) \times S$ is a solution to Eq. (8) in the required class. We now show that there is no more than one such solution. Assume that $f_1(t, x)$ and $f_2(t, x)$ are two solutions from this class. Put $\hat{f}(t, x) = f_1(t, x) - f_2(t, x)$. Then there exists a function $H\hat{f}$, and the relation

$$(16) \quad \lambda \hat{f}(t, x) = \left[\frac{1 + q(x)}{2} \frac{\partial H\hat{f}(t, x+)}{\partial N(x)} - \frac{1 - q(x)}{2} \frac{\partial H\hat{f}(t, x-)}{\partial N(x)} \right]$$

is fulfilled for $t > 0$, $x \in S$. Choose $T_0 > 0$ such that, for all $t \geq T_0$, $x \in S$, $f_1(t, x) = f_2(t, x) = 0$. If $\inf_{t \in [0, T_0], x \in S} H\hat{f}(t, x) = \beta < 0$, then, according to the maximum principle, there exists a point $(t_0, x_0) \in [0, T_0] \times S$ such that $H\hat{f}(t_0, x_0) = \hat{f}(t_0, x_0) = \beta$. This yields the inequalities

$$(17) \quad \frac{\partial H\hat{f}(t_0, x_0-)}{\partial N(x_0)} \leq 0, \quad \frac{\partial H\hat{f}(t_0, x_0+)}{\partial N(x_0)} \geq 0.$$

From (17), we have that the right-hand side of (16) is non-negative at the point (t_0, x_0) . But this contradicts the assertion that $\hat{f}(t_0, x_0) < 0$. Thus, $\inf_{t \in [0, T_0], x \in S} \hat{f}(t, x) = \inf_{t \in [0, T_0], x \in \mathbb{R}^d} H\hat{f}(t, x) \geq 0$. We can get that $\sup_{t \in [0, T_0], x \in S} \hat{f}(t, x) \leq 0$ in the same manner. So, $\hat{f}(t, x) \equiv 0$ on $(t, x) \in [0, \infty) \times S$. This completes the proof for r being equal to 0. In the case of non-negative r , we get the assertion from the previous one arguing as in [4, pp. 194-196].

Proposition 4. *Let r and q be given continuous real-valued functions on S such that r is bounded and non-negative, $|q| \leq 1$. Then the uniqueness theorem is valid for the boundary process.*

Proof. We follow the proof of Theorem 5.2 in [4]. The martingale property can be easily extended to continuous functions on $[0, \infty) \times S$ having compact supports with respect to t in the manner of Remark 2. Given $x \in S$, for any measure \mathbb{R}_x on $\mathcal{D}([0, +\infty), [0, +\infty) \times S)$ being a solution to the submartingale problem starting from x , we have the relation

$$\mathbb{E}^{\mathbb{R}_x}[f(\tau(\theta), y(\theta))] = f(0, x) + \mathbb{E}^{\mathbb{R}_x} \left[\int_0^\theta (\tilde{K}f)(\tau(u), y(u)) du \right].$$

Performing the Laplace transformation, we get, for $\lambda > 0$, the equality

$$\int_0^\infty e^{-\lambda u} \mathbb{E}^{\mathbb{R}_x} [\lambda f(\tau(u), x(\tau(u))) - (\tilde{K}f)(\tau(u), x(\tau(u)))] du = f(0, x).$$

Then, for $\psi \in C_0^\infty([0, \infty) \times S)$, the integral $\int_0^\infty e^{-\lambda u} \mathbb{E}^{\mathbb{R}_x} \psi(\tau(u), x(\tau(u))) du$ is uniquely determined, provided the equation $\lambda f - \tilde{K}f = \psi$ has the unique solution for each $\psi \in C_0^\infty([0, \infty) \times S)$. But this condition is true because of the Lemma. The assertion of the Proposition follows in the way of Corollary 6.2.4 in [9].

3. THE MAIN RESULT

Theorem. *Let S be a closed bounded surface in \mathbb{R}^d which belongs to the class $H^{2+\delta}$ for some $\delta \in (0, 1)$, q and r be given continuous functions on S taking values in $[-1, 1]$ and $[0, +\infty)$, respectively, and r is bounded. For $y \in \mathbb{R}^d$, let $b(y)$ be a symmetric $d \times d$ -matrix satisfying the conditions J . Then, for each $x \in \mathbb{R}^d$, there exists a unique solution to the submartingale problem.*

Proof. This assertion follows from Proposition 5 by the arguments of Theorem 4.2 in [4].

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