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A LIMIT THEOREM FOR SYMMETRIC MARKOVIAN RANDOM EVOLUTION IN $\mathbb{R}^m$

We consider the symmetric Markovian random evolution $\mathbf{X}(t)$ performed by a particle that moves with constant finite speed c in the Euclidean space $\mathbb{R}^m, \ m \geq 2$. Its motion is subject to the control of a homogeneous Poisson process of rate $\lambda > 0$. We show that, under the Kac condition $c \to \infty, \ \lambda \to \infty, \ (c^2/\lambda) \to \rho, \ \rho > 0$, the transition density of $\mathbf{X}(t)$ converges to the transition density of the homogeneous Wiener process with zero drift and the diffusion coefficient $\sigma^2 = 2\rho/m$.

The subject of our interest is the following stochastic motion. A particle starts from the origin $x_1 = \dots = x_m = 0$ of the space $\mathbb{R}^m, \ m \geq 2$, at time $t = 0$. It moves with constant, finite speed c (i.e. c is treated as the constant norm of the velocity). The initial direction is a random m-dimensional vector with uniform distribution (Riemann-Lebesgue probability measure) on the unit sphere

$$S_1^m = \left\{ \mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m : \ x_1^2 + \dots + x_m^2 = 1 \right\}.$$

The particle changes its directions at random instants which form a homogeneous Poisson process of rate $\lambda > 0$. At these moments, it instantaneously takes on the new directions with uniform distribution on S_1^m, independently of its previous motion.

Let $\mathbf{X}(t) = (X_1(t), \dots, X_m(t))$ denote the particle's position at an arbitrary time $t > 0$. At any time $t > 0$, the particle is located with probability 1 in the m-dimensional ball of radius ct:

$$B_{ct}^m = \left\{ \mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m : \ x_1^2 + \dots + x_m^2 \leq c^2 t^2 \right\}.$$

Consider the distribution $Pr \left\{ \mathbf{X}(t) \in d\mathbf{x} \right\}, \ \mathbf{x} \in B_{ct}^m, \ t \geq 0$, where $d\mathbf{x}$ is an infinitesimal volume in the space $\mathbb{R}^m$. This distribution consists of two components. The singular component corresponds to the case where no Poisson event occurs in the interval $(0, t)$ and is concentrated on the sphere

$$S_{ct}^m = \partial B_{ct}^m = \left\{ \mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m : \ x_1^2 + \dots + x_m^2 = c^2 t^2 \right\}.$$

In this case, the particle is located on the sphere S_{ct}^m, and the probability of this event is

$$Pr \left\{ \mathbf{X}(t) \in S_{ct}^m \right\} = e^{-\lambda t}.$$

If one or more than one Poisson events occur, the particle is located strictly inside the ball B_{ct}^m, and the probability of this event is

$$Pr \left\{ \mathbf{X}(t) \in Int \ B_{ct}^m \right\} = 1 - e^{-\lambda t}.$$

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The part of the distribution $Pr\{\mathbf{X}(t) \in d\mathbf{x}\}$ corresponding to this case is concentrated in the interior

$$Int\ B_{ct}^m = \left\{\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m :\ x_1^2 + \dots + x_m^2 < c^2t^2\right\},$$

and forms its absolutely continuous component. Therefore, there exists the density $p(\mathbf{x}, t) = p(x_1, \dots, x_m, t),\ \mathbf{x} \in Int\ B_{ct}^m,\ t > 0$, of the absolutely continuous component of the distribution $Pr\{\mathbf{X}(t) \in d\mathbf{x}\}$.

Our goal is to study the limiting behaviour of $\mathbf{X}(t)$ as both the speed c and the intensity of switches λ tend to infinity.

Consider the characteristic function of the process $\mathbf{X}(t)$ given by

$$H(t) = E\left\{\exp\left(i(\boldsymbol{\alpha}, \mathbf{X}(t))\right)\right\},$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ is the real m-dimensional vector of inversion parameters and $(\boldsymbol{\alpha}, \mathbf{X}(t))$ denotes the inner product of the vectors $\boldsymbol{\alpha}$ and $\mathbf{X}(t)$.

It was shown in [3], [4], [6] that the Laplace transform $\mathcal{L}$ of the characteristic function $H(t)$ has the explicit form

$$\mathcal{L}\left[H(t)\right](s) = \frac{F\left(\frac{1}{2}, \frac{m-2}{2}; \frac{m}{2}; \frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2 + (c\|\boldsymbol{\alpha}\|)^2}\right)}{\sqrt{(s+\lambda)^2 + (c\|\boldsymbol{\alpha}\|)^2} - \lambda\, F\left(\frac{1}{2}, \frac{m-2}{2}; \frac{m}{2}; \frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2 + (c\|\boldsymbol{\alpha}\|)^2}\right)}, \qquad \operatorname{Re} s > 0, \tag{1}$$

where $\|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2 + \dots + \alpha_m^2}$,

$$F(\xi, \eta; \zeta; z) = {}_2F_1(\xi, \eta; \zeta; z) = \sum_{k=0}^{\infty} \frac{(\xi)_k (\eta)_k}{(\zeta)_k} \frac{z^k}{k!}$$

is the Gauss hypergeometric function, and

$$(a)_k = a(a+1)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$$

is the Pochhammer symbol.

We now outline the derivation of formula (1). By using the Markov property and the classical arguments of renewal theory, one can show that the characteristic function $H(t)$ satisfies the Volterra integral equation of the second kind

$$H(t) = e^{-\lambda t}\varphi(t) + \lambda \int_0^t e^{-\lambda(t-\tau)} \varphi(t-\tau) H(\tau)\, d\tau, \qquad t \geq 0,$$

where $\varphi(t)$ is the so-called normalized Bessel function

$$\varphi(t) = 2^{(m-2)/2}\, \Gamma\left(\frac{m}{2}\right) \frac{J_{(m-2)/2}(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^{(m-2)/2}}, \qquad m \geq 2.$$

Note that $\varphi(t)$ is the characteristic function (Fourier transform) of the uniform distribution on the surface of the sphere S_{ct}^m of radius ct. This Volterra integral equation can be rewritten in the convolutional form

$$H(t) = e^{-\lambda t}\varphi(t) + \lambda\left[\left(e^{-\lambda t}\varphi(t)\right) * H(t)\right], \qquad t \geq 0.$$

By applying the Laplace transformation to this equation, we immediately obtain the general form of the Laplace transform of the characteristic function $H(t)$:

$$\mathcal{L}\left[H(t)\right](s) = \frac{\mathcal{L}\left[\varphi(t)\right](s+\lambda)}{1 - \lambda\, \mathcal{L}\left[\varphi(t)\right](s+\lambda)}, \qquad \operatorname{Re} s > 0.$$

By using [1], Table 5.19, formula 6, or [2], formula 6.621(1), we can see that

$$\mathcal{L}\left[\varphi(t)\right](s) = \frac{1}{\sqrt{s^2+(c\|\boldsymbol{\alpha}\|)^2}}\, F\left(\frac{1}{2}, \frac{m-2}{2}; \frac{m}{2}; \frac{(c\|\boldsymbol{\alpha}\|)^2}{s^2+(c\|\boldsymbol{\alpha}\|)^2}\right), \qquad \operatorname{Re} s > 0.$$

Substituting this into the previous expression, we obtain (1).

Formula (1) produces the already known results in lower dimensions. In particular, in the planar case ($m=2$), formula (1) takes the form

$$\mathcal{L}\left[H(t)\right](s) = \frac{1}{\sqrt{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}-\lambda},$$

and this coincides with [9], formula (12) therein. The similar result for the unit speed $c=1$ was given in [13].

In the three-dimensional case ($m=3$), in view of the equality

$$F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{(c\|\boldsymbol{\alpha}\|)^2}{s^2+(c\|\boldsymbol{\alpha}\|)^2}\right) = \frac{\sqrt{s^2+(c\|\boldsymbol{\alpha}\|)^2}}{c\|\boldsymbol{\alpha}\|}\operatorname{arctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{s}\right)$$

which can be easily checked by means of [2], formula 9.121(26), relation (1) immediately yields

$$\mathcal{L}\left[H(t)\right](s) = \frac{\operatorname{arctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)}{c\|\boldsymbol{\alpha}\| - \lambda\operatorname{arctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)}.$$

This exactly coincides with [9], formula (45) therein and, for $c=1$, with [14], formulae (1.6) and (5.8) therein.

Finally, in the four-dimensional case ($m=4$), in view of the equality

$$F\left(\frac{1}{2}, 1; 2; \frac{(c\|\boldsymbol{\alpha}\|)^2}{s^2+(c\|\boldsymbol{\alpha}\|)^2}\right) = \frac{2\sqrt{s^2+(c\|\boldsymbol{\alpha}\|)^2}}{s+\sqrt{s^2+(c\|\boldsymbol{\alpha}\|)^2}}$$

which can be easily checked by means of [2], formula 9.121(24), we obtain from (1) that

$$\mathcal{L}\left[H(t)\right](s) = \frac{2}{s+\sqrt{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}-\lambda}.$$

Our principal result is given by the following theorem.

Limit Theorem. *Under the Kac condition*

$$c\to\infty, \qquad \lambda\to\infty, \qquad \frac{c^2}{\lambda}\to\rho, \qquad \rho>0, \tag{2}$$

the transition density of the process $\mathbf{X}(t)$ *converges to the transition density of the homogeneous Wiener process with zero drift and the diffusion coefficient* $\sigma^2 = 2\rho/m$, *that is,*

$$\lim_{\substack{c,\,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} p(\mathbf{x},t) = \left(\frac{m}{4\rho\pi t}\right)^{m/2} \exp\left(-\frac{m}{4\rho t}\,\|\mathbf{x}\|^2\right), \qquad m\geq 2, \tag{3}$$

where $\|\mathbf{x}\|^2 = x_1^2+\cdots+x_m^2$.

Proof. Our proof consists of three successive steps. First, we will compute the limit of function (1) under the Kac condition (2). Then, by evaluating the inverse Laplace transform, we will find the characteristic function of the limiting process. In the last step, we will compute the inverse Fourier transform of this characteristic function and show that the transition density of the limiting process has the form (3).

It's easy to see that, under the Kac condition (2), we have

$$\lim_{\substack{c,\,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} \frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2} = 0$$

and therefore

$$\lim_{\substack{c,\,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} F\left(\frac{1}{2}, \frac{m-2}{2}; \frac{m}{2}; \frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}\right) = 1.$$

Then by passing to the limit in (1) under the Kac condition (2), we obtain

$$\begin{aligned}
&\lim_{\substack{c,\,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} \mathcal{L}\left[H(t)\right](s) \\
&= \lim_{\substack{c,\,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} \left[\sqrt{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2} - \lambda\, F\left(\frac{1}{2}, \frac{m-2}{2}; \frac{m}{2}; \frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}\right)\right]^{-1} \\
&= \lim_{\substack{c,\,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} \left[(s+\lambda)\sqrt{1+\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)^2} \right. \\
&\qquad\qquad \left. -\lambda \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{m-2}{2}\right)_k}{\left(\frac{m}{2}\right)_k} \frac{1}{k!} \left(\frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}\right)^k\right]^{-1}
\end{aligned} \tag{4}$$

It follows from the Kac condition (2) that, for sufficiently large c and λ, the inequalities

$$\left|\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right| < 1, \qquad \left|\frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}\right| < 1 \tag{5}$$

hold for any s and $\|\boldsymbol{\alpha}\|$. Therefore, the radical in (4) can be represented in the form of the absolutely and uniformly converging series

$$\sqrt{1+\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)^2} = 1 + \frac{1}{2}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)^2 - \frac{1\cdot 1}{2\cdot 4}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)^4 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)^6 - \dots \tag{6}$$

The uniform convergence of series (6) follows from the fact that, under condition (5), it is dominated by the converging numerical series

$$1 + \frac{1}{2} + \frac{1\cdot 1}{2\cdot 4} + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6} + \dots < \infty$$

(see [2], point 1.110 for details).

Similarly, the uniform convergence of the hypergeometric series in (4) follows from the second inequality in (5) and the fact that it is dominated by the converging numerical series

$$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{m-2}{2}\right)_k}{\left(\frac{m}{2}\right)_k} \frac{1}{k!} < \infty.$$

Substituting (6) into (4), we can rewrite it as follows:

$$\begin{aligned}
&\lim_{\substack{c,\,\lambda\to\infty\\(c^2/\lambda)\to\rho}} \mathcal{L}\left[H(t)\right](s)\\
&= \lim_{\substack{c,\,\lambda\to\infty\\(c^2/\lambda)\to\rho}} \Bigg[(s+\lambda)\left(1+\frac{1}{2}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)^2-\frac{1\cdot 1}{2\cdot 4}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)^4+\dots\right)\\
&\qquad -\lambda\Bigg(1+\frac{\left(\frac{1}{2}\right)_1\left(\frac{m-2}{2}\right)_1}{\left(\frac{m}{2}\right)_1}\,\frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}\\
&\qquad\qquad +\frac{1}{2!}\,\frac{\left(\frac{1}{2}\right)_2\left(\frac{m-2}{2}\right)_2}{\left(\frac{m}{2}\right)_2}\left(\frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}\right)^2+\dots\Bigg)\Bigg]^{-1}\\
&= \lim_{\substack{c,\,\lambda\to\infty\\(c^2/\lambda)\to\rho}} \Bigg[s+\lambda+\frac{1}{2}\,\frac{(c\|\boldsymbol{\alpha}\|)^2}{s+\lambda}-\frac{1\cdot 1}{2\cdot 4}\,\frac{(c\|\boldsymbol{\alpha}\|)^4}{(s+\lambda)^3}+\dots\\
&\qquad -\lambda-\frac{\left(\frac{1}{2}\right)_1\left(\frac{m-2}{2}\right)_1}{\left(\frac{m}{2}\right)_1}\,\frac{\lambda(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}\\
&\qquad\qquad -\frac{1}{2!}\,\frac{\left(\frac{1}{2}\right)_2\left(\frac{m-2}{2}\right)_2}{\left(\frac{m}{2}\right)_2}\,\frac{\lambda(c\|\boldsymbol{\alpha}\|)^4}{\left((s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2\right)^2}-\dots\Bigg]^{-1}\\
&= \lim_{\substack{c,\,\lambda\to\infty\\(c^2/\lambda)\to\rho}} \Bigg[s+\frac{1}{2}\,\frac{\frac{c^2}{\lambda}\,\|\boldsymbol{\alpha}\|^2}{\frac{s}{\lambda}+1}-\frac{1\cdot 1}{2\cdot 4}\,\frac{\frac{c^4}{\lambda^3}\,\|\boldsymbol{\alpha}\|^4}{\left(\frac{s}{\lambda}+1\right)^3}+\dots\\
&\qquad -\frac{\left(\frac{1}{2}\right)_1\left(\frac{m-2}{2}\right)_1}{\left(\frac{m}{2}\right)_1}\,\frac{\frac{c^2}{\lambda}\,\|\boldsymbol{\alpha}\|^2}{\left(\frac{s}{\lambda}+1\right)^2+\frac{c^2}{\lambda^2}\,\|\boldsymbol{\alpha}\|^2}\\
&\qquad\qquad -\frac{1}{2!}\,\frac{\left(\frac{1}{2}\right)_2\left(\frac{m-2}{2}\right)_2}{\left(\frac{m}{2}\right)_2}\,\frac{\frac{c^4}{\lambda^3}\,\|\boldsymbol{\alpha}\|^4}{\left(\left(\frac{s}{\lambda}+1\right)^2+\frac{c^2}{\lambda^2}\,\|\boldsymbol{\alpha}\|^2\right)^2}-\dots\Bigg]^{-1}.
\end{aligned}$$

Taking into account both the fact that, under the Kac condition (2), $(c^n/\lambda^{n-1})\to 0$ for any $n\geq 3$ (see also [8], formula (4.4) therein), and the uniform convergence of the series, we obtain

$$\lim_{\substack{c,\,\lambda\to\infty\\(c^2/\lambda)\to\rho}} \mathcal{L}\left[H(t)\right](s)=\left[s+\frac{1}{2}\,\rho\|\boldsymbol{\alpha}\|^2-\frac{\left(\frac{1}{2}\right)_1\left(\frac{m-2}{2}\right)_1}{\left(\frac{m}{2}\right)_1}\,\rho\|\boldsymbol{\alpha}\|^2\right]^{-1}.$$

It's easy to check that

$$\frac{\left(\frac{1}{2}\right)_1\left(\frac{m-2}{2}\right)_1}{\left(\frac{m}{2}\right)_1}=\frac{m-2}{2m},\qquad m\geq 2.$$

Thus, we finally obtain

$$\lim_{\substack{c,\,\lambda\to\infty\\(c^2/\lambda)\to\rho}} \mathcal{L}\left[H(t)\right](s)=\left(s+\frac{\rho\|\boldsymbol{\alpha}\|^2}{m}\right)^{-1}. \tag{7}$$

The inverse Laplace transformation of the function on the right-hand side of (7) yields

$$\mathcal{L}^{-1}\left[\left(s+\frac{\rho\|\boldsymbol{\alpha}\|^2}{m}\right)^{-1}\right](t)=\exp\left(-\frac{\rho\|\boldsymbol{\alpha}\|^2}{m}\,t\right), \tag{8}$$

where we have used [1], Table 5.2, formula 1. The function on the right-hand side of (8) is the characteristic function of the m-dimensional homogeneous Wiener process with zero drift and the diffusion coefficient $\sigma^2=2\rho/m$.

It remains to compute the inverse Fourier transform $\mathcal{F}^{-1}$ of the function on the right-hand side of (8). By applying the Hankel inversion formula (see [15], Chapter 5, Section 23, page 359, formula (43)), we obtain

$$\begin{aligned}\mathbf{w}(\mathbf{x},t) &:= \mathcal{F}^{-1}\left[e^{-(\rho\|\boldsymbol{\alpha}\|^2 t)/m}\right]\\ &= \frac{1}{(2\pi)^{m/2}\|\mathbf{x}\|^{(m-2)/2}}\int_0^\infty e^{-(\rho t/m)\xi^2}\,\xi^{m/2}\,J_{(m-2)/2}(\|\mathbf{x}\|\xi)\,d\xi\\ &= \frac{1}{(2\pi)^{m/2}\|\mathbf{x}\|^{(m-2)/2}}\,\frac{\|\mathbf{x}\|^{(m-2)/2}}{\left(\frac{2\rho t}{m}\right)^{m/2}}\,\exp\left(-\frac{\|\mathbf{x}\|^2}{\frac{4\rho t}{m}}\right)\\ &= \left(\frac{m}{4\rho\pi t}\right)^{m/2}\exp\left(-\frac{m\|\mathbf{x}\|^2}{4\rho t}\right),\end{aligned}$$

proving (3). In the last step, we have used [2], formula 6.631(4), and $J_{(m-2)/2}(x)$ is the Bessel function of the order of $(m-2)/2$ with real argument. The theorem is completely proved.

Function (3) is exactly the transition density of the m-dimensional homogeneous Wiener process with zero drift and the diffusion coefficient $\sigma^2=2\rho/m$. This entirely accords with some previous limiting results for random evolutions (see, for instance, [10], p. 353, Theorem; [11], Proposition 4.8; [12], p. 102, Theorem 4.2.5).

Remark 1. It's easy to see that if $m=2$ and $\rho=1$, density (3) turns into the transition density of the two-dimensional standard Wiener process with zero drift and the diffusion coefficient $\sigma^2=1$ (see [7], p.1181)). For $m=4$ and $\rho=1$, density (3) turns into the transition density of the four-dimensional Wiener process with zero drift and the diffusion coefficient $\sigma^2=1/2$ (see [5], formula (21) therein). It's interesting to note also that if we set $\rho=m/2$, the limiting process becomes the m-dimensional standard homogeneous Wiener process with zero drift and the diffusion coefficient $\sigma^2=1$.

Remark 2. Following [15], Chapter 3, Section 11, Subsection 6, we can easily check that density (3) is the fundamental solution to the m-dimensional heat equation

$$\frac{\partial\mathbf{w}(\mathbf{x},t)}{\partial t}=\frac{\rho}{m}\,\Delta\mathbf{w}(\mathbf{x},t), \tag{9}$$

where Δ denotes the m-dimensional Laplacian. For $\rho=1$, the differential operator on the right-hand side of (9) exactly coincides with the generator obtained in [11], Proposition 4.8.

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