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THE MEASURE PRESERVING AND NONSINGULAR TRANSFORMATIONS OF THE JUMP LÉVY PROCESSES

Let $\xi(t)$, $t \in [0,1]$, be a jump Lévy process. By $\mathcal{P}_\xi$, we denote the law of ξ in the Skorokhod space $\mathbb{D}[0,1]$. Under some conditions on the Lévy measure of the process, we construct the group of $\mathcal{P}_\xi-$ preserving transformations of $\mathbb{D}[0,1]$. For the Lévy process that has only positive (or only negative) jumps, we construct the semigroup of nonsingular transformations.

1. Introduction.

Let $\xi(t)$, $t \in [0,1]$, be a jump Lévy process. It is well known (see [5]) that trajectories of this process with probability 1 belong to the space $\mathbb{D}[0,1]$ of right-continuous functions from $[0,1]$ into $\mathbb{R}$ with left limits. We equip $\mathbb{D}[0,1]$ with the Skorokhod topology and denote, by $\mathcal{P}_\xi$, the law of ξ in $\mathbb{D}[0,1]$.

We use the Lévy-Khinchin representation of these processes ([5]), namely,

$$\xi(t) = at + \int_0^t \int_{0<|x|\leqslant 1} x\widetilde{\nu}(ds,dx) + \int_0^t \int_{|x|>1} x\nu(ds,dx). \tag{1}$$

In this representation, a is a nonrandom constant, $\nu(ds,dx)$ is a Poisson random measure on the space $[0,1]\times\mathbb{R}$ with the intensity measure Π of the form $d\Pi = dt\Lambda(dx)$, where $\Lambda(dx)$ is the Lévy measure of the process ξ. By $\widetilde{\nu}(dt,dx) = \nu(dt,dx) - \mathbb{E}\nu(dt,dx)$, we denote the corresponding compensated measure.

It is well known that the Lévy measure Λ satisfies the following conditions:

1)$\Lambda(\{0\}) = 0$,

2)$\int_{\mathbb{R}} \min(x^2,1)\Lambda(dx) < \infty$.

We also suppose that $\Lambda(\mathbb{R}) = \infty$, that is, Λ is the Lévy measure of a "non-trivial" (i.e., non-compound Poisson) process, and that $\Lambda(\{a\}) = 0$ for any $a \in \mathbb{R}$.

As a probability space for the Poisson random measure, we choose the space of configurations on the set $[0,1]\times\mathbb{R}$ (in a special case of one-sided processes, we use the set $[0,1]\times(0,\infty)$ for this purpose). On this space of configurations, we consider the Poisson measure (see [3]) P with intensity measure Π of the form $\Pi(dt,dx) = dt\Lambda(dx)$.

We consider the distribution of the process ξ in the space $\mathbb{D}[0,1]$ as an image of the Poisson measure P under the action of a mapping defined by (1). First, we consider more general problems.

1) We construct a semigroup of nonsingular (with respect to the Poisson measure P) transformations of the configuration space $\mathcal{X}(G)$, where G is of the form $G = S \times (0,\infty)$

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and S is a complete separable metric space. We suppose that the intensity measure Π of the Poisson measure P is of the form

$$\Pi(d\theta, dx) = \pi(d\theta)\gamma_\theta(dx), \tag{2}$$

where π is a finite measure on S and, for every $\theta \in S$, γ_θ is a $\sigma-$finite measure on $(0,\infty)$ (depending on θ). We suppose that, for every $\varepsilon > 0$, $\Pi(S \times (\varepsilon, \infty)) = \int_S \gamma_\theta((\varepsilon,\infty))\pi(d\theta) < \infty$.

2) We construct a group of $P-$preserving transformations of the configuration space $\mathcal{X}(G)$, where G is of the form $G = S \times \overline{\mathbb{R}}$, (here, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$), and the intensity measure Π of the Poisson measure P is of the form (2) (here, γ_θ is a $\sigma-$finite measure on $\overline{\mathbb{R}}$). We suppose that, for every $\varepsilon > 0$, $\Pi(S \times (\overline{\mathbb{R}} \setminus [-\varepsilon, \varepsilon])) < \infty$.

We consider the transformations $F : \mathcal{X}(G) \to \mathcal{X}(G)$ generated by a mapping $\varphi : G \to G$ such that F maps each configuration $X = \{\overline{x}\}_{\overline{x}\in X}$ to a configuration $Y = F(X)$ of the form

$$F(X) = \{\varphi(\overline{x})\}_{\overline{x}\in X}. \tag{3}$$

It is easy to show that an image of a Poisson measure with intensity measure Π under the action of (3) is a Poisson measure with intensity measure $\Pi\varphi^{-1}$. The absolute continuity conditions for Poisson measures with different intensity measures were first obtained by Skorokhod [4] (see also [1,3,7]).

Later on, Vershik and Tsilevich [8] considered the absolute continuous transformations of the so-called gamma measure which is a Poisson measure with the intensity measure Π of the form

$$\Pi(d\theta, dx) = \pi(d\theta)\frac{e^{-x}}{x}dx,\ \theta \in S,\ x \in (0,\infty).$$

(Here as above, S is a complete separable metric space, and π is a finite measure on S.) It was proved in [8] that the gamma measure is quasi-invariant under a group of transformations $F_a,\ a \in \mathcal{M}$, where $\mathcal{M}$ is a set of all measurable functions $a : S \to (0,\infty)$ such that

$$\int_S |\log a(\theta)|\pi(d\theta) < \infty,$$

and F_a is a transformation of the form (3) with $\varphi(\theta, x) = (\theta, a(\theta)x)$. Note that the set $\mathcal{M}$ is a commutative group with respect to the pointwise multiplication of functions and, for every $a_1, a_2 \in \mathcal{M}$, we have $F_{a_1 a_2} = F_{a_1} \circ F_{a_2}$.

In [6], the transformations of the so-called stable measures P_α, $\alpha > 0$, which are the Poisson measures with intensity measures of the form

$$\pi(d\theta)\frac{dx}{x^{1+\alpha}},\ \theta \in S,\ x \in (0,\infty)$$

were considered. It was proved there that, for every $\alpha > 0$, a stable measure P_α is quasi-invariant under a semigroup of transformations $\Phi_f,\ f \geq 0,\ f \in L_1(S,\pi)$, where Φ_f is a transformation of the form (3) with

$$\varphi(\theta, x) = \left(\theta, \frac{x}{(1 + \alpha f(\theta)x^\alpha)^{1/\alpha}}\right). \tag{4}$$

Further, in the same paper, it was proved that, for every $\alpha > 0$, the Poisson measure $\widetilde{P_\alpha}$ on the configuration space on $S \times \overline{\mathbb{R}}$ (here, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$) with intensity measure $\pi(d\theta)\frac{dx}{|x|^{1+\alpha}}$ is invariant under a group of transformations $\widetilde{\Phi}_f$, where f is an arbitrary measurable function on S, and $\widetilde{\Phi}_f$ is a transformation of the form (3) with the function φ defined by (4) [in (4), the functions x^α and $x^{1/\alpha}$ are extended to the whole real line in an odd way].

In the present paper, the similar results are proved for an arbitrary Poisson measure with an intensity measure of the form (2).

The paper is organized as follows. Secton 2 contains the necessary definitions concerning the configuration space and Poisson measures. In Section 3, we construct a semigroup of nonsingular (with respect to the Poisson measure P) transformations of the configuration space $\mathcal{X}(G)$, where G is of the form $S\times(0,\infty)$, and the intensity measure of the Poisson measure P is of the form (2). In Section 4, we construct a group of $P-$preserving transformations in the case where $G=S\times\overline{\mathbb{R}}$. In Section 5, we construct the groups and the semigroups of transformations of trajectories of the Lévy process. Finally, in Section 6, we consider one multidimensional generalization of the results of Section 4.

2. The space of configurations.

Let G be a metric space, $\mathcal{B}$ be its Borel $\sigma-$algebra, $\mathcal{B}_0$ be the ring of bounded Borel subsets of G. Let Π be a $\sigma-$finite measure on G. Suppose that $\Pi(V)<\infty$ for every $V\subset\mathcal{B}_0$.

We denote, by $\mathcal{X}=\mathcal{X}(G)$, the space of configurations on G. By definition,

$$\mathcal{X}(G)=\{X\subset G:|X\cap V|<\infty \text{ for all } V\subset\mathcal{B}_0\},$$

where $|A|$ denotes the cardinality of the set A. We equip $\mathcal{X}$ with the vague topology $\mathcal{O}(\mathcal{X})$, i.e., the weakest topology such that all functions $\mathcal{X}\to\mathbb{R}$

$$X\mapsto\sum_{x\in X}f(x)$$

are continuous for all continuous functions $f:G\to\mathbb{R}$ with bounded supports. The Borel $\sigma-$algebra corresponding to $\mathcal{O}(\mathcal{X})$ will be denoted by $\mathcal{B}(\mathcal{X})$. This is the smallest $\sigma-$algebra for which the mapping

$$X\mapsto|X\cap V|$$

is measurable for any $V\subset\mathcal{B}_0$.

We say that a probability measure P on $(\mathcal{X},\mathcal{B}(\mathcal{X}))$ is the Poisson measure with intensity measure Π, if, for every $V\in\mathcal{B}_0$,

$$P(X:|X\cap V|=k)=e^{-\Pi(V)}\frac{\Pi(V)^k}{k!}.$$

For more details, see, e.g., [2,3].

3. Nonsingular transformations of the Poisson measure.

In this section, we suppose that $G=S\times(0,\infty)$, where S is a complete separable metric space, and the measure Π is of the form

$$\Pi(d\theta,dx)=\pi(d\theta)\gamma_\theta(dx),\ \theta\in S,\ x\in(0,\infty),$$

where π is a finite measure on S. We also suppose that, for $\pi-$a.e. $\theta\in S$,

1) $\gamma_\theta((0,\infty))=\infty$,

2) for every $x>0$ $\gamma_\theta((x,\infty))<\infty$, and the function

$$U_\theta(x)=\gamma_\theta((x,\infty)) \tag{5}$$

is continuous and strictly decreasing. Set $U_\theta(0)=\infty,\ U_\theta(\infty)=0$.

Further, for $\theta\in S,\ t\geqslant 0$, by T_t^θ, we denote a mapping from $(0,\infty)$ into $(0,\infty)$ defined by the formula

$$T_t^\theta(x)=U_\theta^{-1}(U_\theta(x)+t). \tag{6}$$

Note that T_0^θ is the identity mapping. Let us show that, for any $s, t \in [0, \infty)$,

$$T_{t+s}^\theta = T_t^\theta \circ T_s^\theta. \tag{7}$$

Using (6), we get

$$T_t^\theta \circ T_s^\theta(x) = T_t^\theta(U_\theta^{-1}(U_\theta(x) + s)) = U_\theta^{-1}(U_\theta(U_\theta^{-1}(U_\theta(x) + s)) + t) =$$

$$U_\theta^{-1}(U_\theta(x) + s + t) = T_{t+s}^\theta.$$

Denote, by $L^+(S)$, the space of all Borel nonnegative functions on S, and, by $L_1^+(S, \pi)$, $L_1^+(S, \pi) \subset L^+(S)$, the space of all nonnegative integrable functions with respect to the measure π.

For $f \in L^+(S)$, we define a mapping $\tau_f : G \to G$ by

$$\tau_f(\theta, x) = (\theta, T_{f(\theta)}^\theta(x)). \tag{8}$$

Note that, for $f \equiv 0$, the mapping τ_f is the identity mapping. It follows easily from (7) that, for every $f, g \in L^+(S)$,

$$\tau_{f+g} = \tau_f \circ \tau_g. \tag{9}$$

Now, using the semigroup of transformations $\tau_f,\ f \in L^+(S)$, we construct a semigroup $\Phi_f,\ f \in L^+(S)$ of transformations of $\mathcal{X}(S \times (0, \infty))$. In accordance with (3), for $f \in L^+(S)$, we define a mapping $\Phi_f : \mathcal{X}(S \times (0, \infty)) \to \mathcal{X}(S \times (0, \infty))$ as a mapping generated by τ_f,

$$\Phi_f(X) = Y = \{(\theta, T_{f(\theta)}^\theta(x))\}. \tag{10}$$

It follows from (7) and (8) that, for every $f, g \in L^+(S)$,

$$\Phi_{f+g} = \Phi_f \circ \Phi_g. \tag{11}$$

Note that, in the case where the measure $\gamma_\theta(dx)$ has the form $\frac{dx}{x^{1+\alpha}},\ \alpha > 0$, we get, for every $\theta \in S$,

$$\tau_f(\theta, x) = \left(\theta, \frac{x}{(1 + \alpha f(\theta) x^\alpha)^{1/\alpha}}\right),$$

that corresponds to the semigroup of transformations constructed in [6].

The main result of this section is the following.

Theorem 1. *1. The measure $P\Phi_f^{-1}$ is absolutely continuous with respect to P for every $f \in L_1^+(S, \pi)$ and has the form*

$$P\Phi_f^{-1} = e^{\int_S f\, d\pi} \cdot P\big|_{A_f},$$

where A_f is a measurable subset of $\mathcal{X}(S \times (0, \infty))$, Φ_f maps the measure P to its restriction $P\big|_{A_f}$ on A_f, $e^{\int_S f\, d\pi} = \frac{1}{P(A_f)}$ is a natural normalizing coefficient.

2. For every $f \in L^+(S)$ such that $\int_S f\, d\pi = \infty$, the measures $P\Phi_f^{-1}$ and P are orthogonal.

Proof. First, for fixed $\theta \in S,\ t > 0$, we calculate the image of the measure γ_θ under the action of $T_t^\theta : (0, \infty) \to (0, \infty)$.

It is easy to check that $T_t^\theta(x)$ is a strictly increasing function of the variable x, and it follows from the equality $T_t^\theta(+\infty) = U_\theta^{-1}(t)$ that the support of the measure $\gamma_\theta(T_t^\theta)^{-1}$ is an interval $(0, U_\theta^{-1}(t))$. We now show that, on this interval, the measure $\gamma_\theta(T_t^\theta)^{-1}$ coincides with the measure γ, i.e.,

$$\gamma_\theta(T_t^\theta)^{-1} = \gamma_\theta\big|_{(0, U_\theta^{-1}(t))}. \tag{12}$$

For an arbitrary interval $[\alpha,\beta]\subset(0,U_\theta^{-1}(t))$, we have

$$\gamma_\theta(T_t^\theta)^{-1}([\alpha,\beta])=\gamma_\theta([(T_t^\theta)^{-1}(\alpha),(T_t^\theta)^{-1}(\beta)])=$$
$$U_\theta(U_\theta^{-1}(U_\theta(\alpha)-t))-U_\theta(U_\theta^{-1}(U_\theta(\beta)-t))=$$
$$U_\theta(\alpha)-U_\theta(\beta)=\gamma_\theta([\alpha,\beta]).$$

This completes the proof of (12). We deduce from (12) that the image of the measure $\Pi(d\theta,dx)=\pi(d\theta)\gamma_\theta(dx)$ under the action of τ_f, is a measure of the form

$$\Pi\tau_f^{-1}(d\theta,dx)=\mathbf{1}_{(0,U_\theta^{-1}(f(\theta)))}(x)\pi(d\theta)\gamma_\theta(dx), \tag{13}$$

where $\mathbf{1}_B$ denotes the indicator function of the set B. Consider a subset G_f of the set $G=S\times(0,\infty)$ such that the density $\frac{d\Pi\tau_f^{-1}}{d\Pi}$ is equal to 1 on G_f, i.e.,

$$G_f=\{(\theta,x)\in G:0<x<U_\theta^{-1}(f(\theta))\}. \tag{14}$$

It follows from (13) that the measure $P\Phi^{-1}$ is concentrated on a set A_f of the form

$$A_f=\{X\in\mathcal{X}(G):X\cap(G\setminus G_f)=\emptyset\}.$$

Note that A_f coincides with the space $\mathcal{X}(G_f)$ of configurations on the space G_f. By (13), we know that the intensity measure of the Poisson measure $P\Phi_f^{-1}$ restricted on G_f is equal to $\pi(d\theta)\gamma_\theta(dx)$.

Hence, the measure PT_f^{-1} is absolutely continuous with respect to P if $P(A_f)>0$ and PT_f^{-1} is orthogonal to P if $P(A_f)=0$. To complete the proof, it remains to calculate $P(A_f)$.

We have

$$P(A_f)=P(X:\mid X\cap(G\setminus G_f)\mid=0)=\exp(-\Pi(G\setminus G_f))=$$
$$\exp(-\int_S\pi(d\theta)\int_{U_\theta^{-1}(f(\theta))}^\infty\gamma_\theta(dx))=\exp(-\int_S f\,d\pi).$$

The last expression is positive iff $\int_S f\,d\pi<\infty$. This completes the proof of the theorem.

4. The measure preserving transformations.

In this section, we suppose that the set $G=S\times\overline{\mathbb{R}}$, where $\overline{\mathbb{R}}=\mathbb{R}\cup\{\infty\}$ is an extended real line (by definition, the point ∞ has no sign, so that $-\infty=+\infty$). We also suppose that the intensity measure Π of the Poisson measure P is of the form

$$\Pi(d\theta,dx)=\pi(d\theta)\gamma_\theta(dx),$$

where π is a finite measure on S, and, for every $\theta\in S$, γ_θ is a $\sigma-$finite measure on $\overline{\mathbb{R}}$ (depending on θ).

For every fixed $\theta\in S$, let $U_\theta:\overline{\mathbb{R}}\to\overline{\mathbb{R}}$ denote a function defined by

$$U_\theta(x)=\begin{cases}\gamma_\theta((x,\infty)), & \text{if } x>0;\\ -\gamma_\theta((-\infty,x)), & \text{if } x<0.\end{cases} \tag{15}$$

Set $U_\theta(\infty)=0,\ U_\theta(0)=\infty$. Note that, for $x>0$, formula (15) coincides with (5).

We suppose that, for $\pi-$a.e. $\theta\in S$,

1) $\gamma_\theta(\{0\})=\gamma_\theta(\{\infty\})=0$,

2) $\gamma_\theta((0,\infty))=\gamma_\theta((-\infty,0))=+\infty$,

3) for every $\varepsilon>0$ $\gamma_\theta(\overline{\mathbb{R}}\setminus[-\varepsilon,\varepsilon])<\infty$,

4) U_θ is continuous on $\overline{\mathbb{R}}\setminus\{0\}$ and strictly decreasing on $(-\infty,0)$ and $(0,\infty)$.

First, for every fixed $\theta\in S$, we construct a one-parameter group of $\gamma_\theta-$preserving transformations.

For $t \in (-\infty, \infty)$, let T_t^θ be a map from $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$ such that

$$T_t^\theta(x) = U_\theta^{-1}\big(U_\theta(x) + t\big). \tag{16}$$

We preserve here the same notation T_t^θ as in (6). We only note that, unlike (6), the domain of mapping (16) is $\overline{\mathbb{R}}$, and the parameter t may be either positive or negative.

It is easily proved that $T_t^\theta,\ t \in (-\infty, \infty)$, is a one-parameter group of transformations, i.e., for any $s,\, t \in \mathbb{R}$.

$$T_{t+s}^\theta = T_t^\theta \circ T_s^\theta. \tag{17}$$

Show that the group $T_t^\theta,\ t \in (-\infty, \infty)$, is a group of $\gamma_\theta-$preserving transformations, i.e., for every $t \in (-\infty, \infty)$,

$$\gamma_\theta (T_t^\theta)^{-1} = \gamma_\theta. \tag{18}$$

Notice that T_t^θ is a superposition of three transformations, namely, U_θ, a shift transformation, and U_θ^{-1}. The first transformation maps the measure γ_θ into a Lebesgue measure, the shift transformation preserves the Lebesgue measure, and U_θ^{-1} maps the Lebesgue measure into γ_θ.

As before, we denote the set of all Borel functions on S by $L(S)$. Given $f \in L(S)$, we define a mapping

$$\tau_f : S \times \overline{\mathbb{R}} \to S \times \overline{\mathbb{R}},$$

by

$$\tau_f(\theta, x) = (\theta, T_{f(\theta)}^\theta). \tag{19}$$

It follows from (17) that, for every $f, g \in L(S)$,

$$\tau_{f+g} = \tau_f \circ \tau_g, \tag{20}$$

and it follows from (18) that, for every $f \in L(S)$,

$$\Pi \tau_f^{-1} = \Pi.$$

Using the group of transformations $\tau_f,\ f \in L(S)$, we construct a group $\widetilde{\Phi}_f,\ f \in L(S)$ of transformations of $\mathcal{X}(S \times \overline{\mathbb{R}})$ by analogy with (10). In accordance with (3), for $f \in L(S)$, we define a mapping $\widetilde{\Phi}_f : \mathcal{X}(S \times \overline{\mathbb{R}}) \to \mathcal{X}(S \times \overline{\mathbb{R}})$ as a mapping generated by τ_f and such that

$$\widetilde{\Phi}_f(X) = Y = \{(\theta, T_{f(\theta)}^\theta(x))\}.$$

The main result of this section is the following

Theorem 2.

1. For every $f, g \in L(S)$, we have

$$\widetilde{\Phi}_{f+g} = \widetilde{\Phi}_f \circ \widetilde{\Phi}_g.$$

2. For every $f \in L(S)$,

$$P\widetilde{\Phi}_f^{-1} = P,$$

i.e., the group $\widetilde{\Phi}_f,\ f \in L(S)$ is a group of $P-$preserving transformations.

Proof. Statement 1 of the theorem follows from (20). To prove statement 2, we note that the measure $P\widetilde{\Phi}_f^{-1}$ is the Poisson measure with intensity measure $\Pi\tau_f^{-1} = \Pi$.

5. The transformations of the trajectories of Lévy processes.

First we consider the Lévy process that has only positive jumps. The Lévy measure Λ of such a process is concentrated on $(0,\infty)$. Suppose that

1) $\Lambda((0,\infty))=\infty$,

2) the function

$$U(x)=\Lambda((x,\infty)) \tag{21}$$

is continuous and strictly decreasing on $(0,\infty)$.

Given $G=[0,1]\times(0,\infty)$, consider the space of configurations $\mathcal{X}(G)$ and the Poisson measure P with intensity measure $\Pi=\lambda\times\Lambda$, where λ is a Lebesgue measure on $[0,1]$.

In accordance with (1), a Lévy process ξ with the Lévy measure Λ can be defined on the probability space $(\mathcal{X}(G),\mathcal{B}(\mathcal{X}(G)),P)$ by

$$\xi(t,X)=at+(L_2)\lim_{\varepsilon\to 0}\Bigg(\sum_{\substack{(s,x)\in X,\\ s\leqslant t,\varepsilon\leqslant x\leqslant 1}} x-t\int_\varepsilon^1 x\Lambda(dx)\Bigg)+\sum_{\substack{(s,x)\in X,\\ s\leqslant t,1<x<\infty}} x. \tag{22}$$

The sum is taken over all points (s,x) of configurations $X\in\mathcal{X}$. The relation between the configurations and the corresponding trajectories of the random process is very simple. Namely, if the point (s,x) belongs to the configuration X, then the corresponding trajectory $\xi(\cdot,X)$ of the random process ξ at the moment s has a jump which is equal to x. We denote, by Ξ, the mapping $\mathcal{X}\to\mathbb{D}[0,1]$ defined by (22) so that

$$\Xi(X)=\xi(\cdot,X).$$

Note that the different configurations generate different trajectories of a random process. Namely, if $X_1\neq X_2$, then

$$\Xi(X_1)\neq\Xi(X_2). \tag{23}$$

Let $\mathcal{P}_\xi$ denote a measure generated by ξ in the space $\mathbb{D}[0,1]$, i.e.,

$$\mathcal{P}_\xi=P\Xi^{-1}. \tag{24}$$

Given a function $z\in\mathbb{D}[0,1]$, we denote, by Δz, the function

$$\Delta z(t)=z(t)-z(t-0),t\in[0,1].$$

Further for $f\in L_1^+[0,1]$, we denote a mapping $\Psi_f:\mathbb{D}[0,1]\to\mathbb{D}[0,1]$, by

$$[\Psi_f(z)](t)=z(t)+\sum_{s\leqslant t}\big(T_{f(s)}(\Delta z(s))-\Delta z(s)\big), \tag{25}$$

where $T_t,t\geqslant 0$ is a semigroup of transformations defined by (6), i.e., $T_t(x)=U^{-1}(U(x)+t)$. Here, U is defined by (21) and, unlike the general case considered in (6), it does not depend on $t\in[0,1]$.

Further, we consider the semigroup of nonsingular transformations

$$\Phi_f,\qquad f\in L_1^+([0,1],\lambda),$$

of the configuration space $\mathcal{X}(G)$, $G=[0,1]\times(0,\infty)$ defined by (10). It follows from (10), (22), and (25) that

$$\Psi_f\circ\Xi=\Xi\circ\Phi_f. \tag{26}$$

Now it follows from (26) that the mapping Ψ_f is correctly defined by (25) $\mathcal{P}_\xi-$a.s.. Moreover, it follows from (7) and (26) that, for every $f,g\in L_1^+([0,1],\lambda)$,

$$\Psi_{f+g}=\Psi_f\circ\Psi_g.$$

Further, for $f \in L_1^+([0,1],\lambda)$, we define a subset $\mathbb{D}_f$ of the set $\mathbb{D}[0,1]$ by

$$\mathbb{D}_f = \{z \in \mathbb{D}[0,1] : \forall t \in (0,1] \quad 0 \leqslant \Delta z(t) < U^{-1}(f(t))\}.$$

Theorem 3. *For every* $f \in L_1^+([0,1],\lambda)$,

$$\mathcal{P}_\xi \Psi_f^{-1} \ll \mathcal{P}_\xi,$$

and, moreover,

$$\mathcal{P}_\xi \Psi_f^{-1} = e^{\int_0^1 f(t)dt} \cdot \mathcal{P}_\xi\big|_{\mathbb{D}_f}.$$

Proof. Using Theorem 1 and (26), we get

$$P\Phi_f^{-1} \ll P,$$

therefore,

$$(P\Phi_f^{-1})\Xi^{-1} \ll P\Xi^{-1}. \tag{27}$$

The RHS of (27) equals $\mathcal{P}_\xi$. Now we calculate the LHS of (27):

$$(P\Phi_f^{-1})\Xi^{-1} = P(\Xi \circ \Phi_f)^{-1} = P(\Psi_f \circ \Xi)^{-1} = (P\Xi^{-1})\Psi_f^{-1} = \mathcal{P}_\xi \Psi_f^{-1}.$$

Consider a Lévy process with both positive and negative jumps. The Lévy measure Λ of such a process satisfies the conditions $\Lambda((0,\infty)) > 0$ and $\Lambda((-\infty,0)) > 0$. We suppose in addition that the measure Λ satisfies the conditions

1) $\Lambda((0,\infty)) = \infty$, and $\Lambda((-\infty,0)) = \infty$,

2) the function $U : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ defined by the formula

$$U(x) = \begin{cases} \Lambda((x,\infty)), & \text{if } x > 0; \\ -\Lambda((-\infty,x)), & \text{if } x < 0 \end{cases}$$

is continuous and strictly decreasing on the intervals $(0,\infty)$ and $(-\infty,0)$.

On the space of configurations $\mathcal{X}([0,1]\times\overline{\mathbb{R}})$, we consider the Poisson measure P with intensity measure $\Pi(dt,dx) = dt\Lambda(dx)$.

In accordance with (1), the Lévy process $\xi(t)$ with the Lévy measure Λ can be defined on the probability space $(\mathcal{X}([0,1]\times\overline{\mathbb{R}}), P)$ by

$$\xi(t,X) = at + (L_2)\lim_{\varepsilon\to 0}\Bigg(\sum_{\substack{(s,x)\in X,\\ s\leqslant t, \varepsilon\leqslant|x|\leqslant 1}} x - t\int_{\varepsilon\leqslant|x|\leqslant 1} xd\Lambda\Bigg) + \sum_{\substack{(s,x)\in X,\\ s\leqslant t, |x|>1}} x. \tag{28}$$

Further, consider the group $T_t, t \in (-\infty,\infty)$ defined by (16) and the corresponding group $\widetilde{\Phi}_f$, $f \in L[0,1]$, of transformations of $\mathcal{X}([0,1]\times\overline{\mathbb{R}})$. As above, using a group $\widetilde{\Phi}_f$ and a mapping Ξ [defined by (28)], we construct a group $\widetilde{\Psi}_f$ of transformations of the space $\mathbb{D}[0,1]$ so that

$$\widetilde{\Psi}_f \circ \Xi = \Xi \circ \widetilde{\Phi}_f.$$

It is easy to prove that the mapping $\widetilde{\Psi}_f$ can be defined by the formula

$$[\widetilde{\Psi}_f(z)](t) = z(t) + \sum_{\substack{s\leqslant t,\\ T_{f(s)}(\Delta z(s))\neq\infty}} (T_{f(s)}(\Delta z(s)) - \Delta z(s)), \tag{29}$$

where

$$\Delta z(t) = z(t) - z(t-0), \ t \in [0,1].$$

Theorem 4. *For every Borel function $f : [0,1] \to \mathbb{R}$, the mapping $\widetilde{\Psi}_f$ is correctly defined $\mathcal{P}_\xi$−a.s. and preserves the measure $\mathcal{P}_\xi$, i.e.,*

$$\mathcal{P}_\xi \widetilde{\Psi}_f^{-1} = \mathcal{P}_\xi.$$

Proof follows from Theorem 3.

6. One multidimensional generalization.

In this section, we consider the configuration space $\mathcal{X}(G)$, where G has the form $S \times (\overline{\mathbb{R}^d} \setminus \{0\})$, S is a complete separable metric space, and $\overline{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty\}$. By S_{d-1}, we denote a unit sphere in $\mathbb{R}^d$ and, by $\sigma(d\omega),\ \omega \in S_{d-1}$, we denote the surface measure on S_{d-1}.

Let Q denote the mapping from $\mathbb{R}^d \setminus \{0\}$ into $S_{d-1} \times (0,\infty)$ defined by

$$Q(x) = (\frac{x}{\|x\|}, \|x\|). \tag{30}$$

It follows from (30) that, for $\omega \in S_{d-1},\ r \in (0,\infty)$,

$$Q^{-1}(\omega, r) = r\omega \in \mathbb{R}^d \setminus \{0\}.$$

Recall that the representation of a measure μ in polar coordinates on $\mathbb{R}^d \setminus \{0\}$ is a measure μQ^{-1} on $S_{d-1} \times (0,\infty)$ (that is, the image of the measure μ under the action of Q).

On the configuration space $\mathcal{X}(G) = \mathcal{X}(S \times (\overline{\mathbb{R}^d} \setminus \{0\}))$, we consider a Poisson measure P with intensity measure Π of the form

$$\Pi(d\theta, dx) = \pi(d\theta)\Gamma_\theta(dx), \tag{31}$$

where, as above, π is a finite measure on S, and Γ_θ is a σ−finite measure on $\overline{\mathbb{R}^d} \setminus \{0\}$. We suppose that $\Gamma_\theta(\infty) = 0$ for π−a.s. $\theta \in S$, and the representation of the measure Γ_θ in polar coordinates has the form

$$\Gamma_\theta Q^{-1}(d\omega, dr) = \sigma(d\omega)\gamma_\omega^\theta(dr),$$

where, in turn, the measure γ_ω^θ satisfies, for σ−a.s. $\omega \in S_{d-1}$, the following conditions:

1) $\gamma_\omega^\theta((0,\infty)) = \infty$,

2) for every $r > 0$ $\gamma_\omega^\theta((r,\infty)) < \infty$, and the function

$$U_\omega^\theta(r) = \gamma_\omega^\theta((r,\infty))$$

is continuous and strictly decreasing.

For fixed $\theta \in S$, we construct a d−parametric group of Γ_θ−preserving transformations of $\overline{\mathbb{R}^d} \setminus \{0\}$. To this end, we define a mapping f_ω^θ from $(0,\infty]$ into $[0,\infty)$ for $\omega \in S_{d-1}$ by the formula

$$f_\omega^\theta(r) = \left(d \cdot U_\omega^\theta(r)\right)^{1/d} \tag{33}$$

for $r \in (0,\infty)$. Set $f_\omega^\theta(\infty) = 0$. It is easy to see that, for $r \in (0,\infty)$,

$$(f_\omega^\theta)^{-1}(r) = \left(U_\omega^\theta\right)^{-1}(\frac{r^d}{d}). \tag{34}$$

Further, by μ, we denote a measure on $[0,\infty)$ of the form $\mu(dr) = r^{d-1}dr$.

Lemma 1. *For $\theta \in S$, $\omega \in S_{d-1}$, the mapping f_ω^θ transforms the measure γ_ω^θ into the measure μ and, conversely, the mapping $(f_\omega^\theta)^{-1}$ transforms the measure μ into γ_ω^θ, i.e.,*

$$\gamma_\omega^\theta (f_\omega^\theta)^{-1} = \mu$$

and

$$\mu f_\omega^\theta = \gamma_\omega^\theta.$$

Proof. We denote the Lebesgue measure on $[0,\infty)$ by λ and the mapping from $[0,\infty)$ into $[0,\infty)$ by g defined by the formula $g(r) = \frac{r^d}{d}$. Note that $f_\omega^\theta = g^{-1} \circ U_\omega^\theta$.

It is easy to check that $\mu g^{-1} = \lambda$ and $\gamma_\omega^\theta (U_\omega^\theta)^{-1} = \lambda$. So, we have

$$\gamma_\omega^\theta (f_\omega^\theta)^{-1} = \gamma_\omega^\theta (g^{-1} \circ U_\omega^\theta)^{-1} = \mu.$$

By F_θ, we denote the mapping from $S_{d-1} \times (0,\infty]$ into $S_{d-1} \times [0,\infty)$ defined by the formula

$$F_\theta(\omega, r) = (\omega, f_\omega^\theta(r)) \tag{35}$$

It follows easily from Lemma 1 that the mapping F_θ transforms the measure $\Gamma_\theta Q^{-1}$ into the measure $\sigma \times \mu$, i.e.,

$$(\Gamma_\theta Q^{-1}) F_\theta^{-1} = \sigma \times \mu. \tag{36}$$

Further, by E_θ, $E_\theta : \overline{\mathbb{R}^d} \setminus \{0\} \to \mathbb{R}^d$, we denote the mapping

$$E_\theta = Q^{-1} \circ F_\theta \circ Q. \tag{37}$$

It follows from (33) and (37) that

$$E_\theta(x) = \frac{x}{\|x\|} \big(d U^\theta_{\frac{x}{\|x\|}} (\|x\|) \big)^{1/d} \tag{38}$$

and

$$E_\theta^{-1}(x) = \frac{x}{\|x\|} \big(U^\theta_{\frac{x}{\|x\|}} \big)^{-1} \big(\frac{\|x\|^d}{d} \big). \tag{39}$$

Now, for every $t \in \mathbb{R}^d$, we define a mapping T_t^θ, $T_t^\theta : \overline{\mathbb{R}^d} \setminus \{0\} \to \overline{\mathbb{R}^d} \setminus \{0\}$ by

$$T_t^\theta(x) = E_\theta^{-1}(E_\theta(x) + t). \tag{40}$$

It is clear that, for every $s, t \in \mathbb{R}^d$,

$$T_{t+s}^\theta = T_t^\theta \circ T_s^\theta. \tag{41}$$

We now show that the group T_t^θ, $t \in \mathbb{R}^d$ is a $d-$parametric group of $\Gamma_\theta-$preserving transformations of $\overline{\mathbb{R}^d} \setminus \{0\}$, i.e., for every $t \in \mathbb{R}^d$,

$$\Gamma_\theta (T_t^\theta)^{-1} = \Gamma_\theta. \tag{42}$$

Notice that T_t^θ is a superposition of three transformations, namely, E_θ, shift transformation, and E_θ^{-1}. It follows from (36) and (37) that the transformation E_θ transforms the measure Γ_θ into the Lebesgue measure, the shift transformation preserves the Lebesgue measure, and U_θ^{-1} transforms the Lebesgue measure into Γ_θ.

By $L(S, \mathbb{R}^d)$, we denote the set of all Borel functions on S taking value in $\mathbb{R}^d$. Given $f \in L(S, \mathbb{R}^d)$, we define a mapping

$$\tau_f : S \times (\overline{\mathbb{R}^d} \setminus \{0\}) \to S \times (\overline{\mathbb{R}^d} \setminus \{0\}),$$

by

$$\tau_f(\theta, x) = (\theta, T^\theta_{f(\theta)}). \tag{43}$$

It follows from (41) that, for every $f, g \in L(S, \mathbb{R}^d)$,

$$\tau_{f+g} = \tau_f \circ \tau_g, \tag{44}$$

and (42) implies that, for every $f \in L(S, \mathbb{R}^d)$,

$$\Pi\tau_f^{-1} = \Pi.$$

Using the group of transformations τ_f, $f \in L(S)$, we construct a group $\widetilde{\Phi}_f$, $f \in L(S, \mathbb{R}^d)$ of transformations of $\mathcal{X}(S \times (\overline{\mathbb{R}^d} \setminus \{0\}))$ by analogy with (10). In accordance with (3), for $f \in L(S, \mathbb{R}^d)$, we define a mapping $\widetilde{\Phi}_f : \mathcal{X}(S \times (\overline{\mathbb{R}^d} \setminus \{0\})) \to \mathcal{X}(S \times (\overline{\mathbb{R}^d} \setminus \{0\}))$ as a mapping generated by τ_f and such that

$$\widetilde{\Phi}_f(X) = Y = \{(\theta, T^{\theta}_{f(\theta)}(x))\}.$$

The main result of this section is the following

Theorem 5.

1. For every $f, g \in L(S, \mathbb{R}^d)$, we have

$$\widetilde{\Phi}_{f+g} = \widetilde{\Phi}_f \circ \widetilde{\Phi}_g.$$

2. For every $f \in L(S, \mathbb{R}^d)$,

$$P\widetilde{\Phi}_f^{-1} = P,$$

i.e., the group $\widetilde{\Phi}_f$, $f \in L(S, \mathbb{R}^d)$, is a group of $P-$ preserving transformations.

Proof. The statements of this theorem can be proved by the same arguments as Theorem 2.

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