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# ASYMPTOTIC FORMULAS FOR PROBABILITIES OF LARGE DEVIATIONS OF LADDER HEIGHTS

Asymptotic formulas for large-deviation probabilities of a ladder height in a random walk generated by a sequence of sums of i.i.d. random variables are deduced.

Two cases are considered:

a) the distribution $F(x)$ of summands is normal with a zero mean.
b) $F(x)$ belongs to the domain of the normal attraction of a stable law with the exponent $0 < \alpha < 1$.

The method of Laplace transforms is applied in proofs.

## 1. Introduction

Let $X, X_1, X_2, \ldots, X_n, \ldots$ be i.i.d. variables with the distribution function $F(x)$, not degenerate at zero. Put

$$S_n = \sum_{i=1}^{n} X_i,\ F_n(D) = \mathbf{P}(S_n \in D).$$

Introduce the notation

$$N^+ = \min\Big\{n : S_n > 0\Big\},\ N^- = \min\Big\{n : S_n \le 0\Big\}.$$

Let $Z_+ := S_{N^+},\ Z_- := S_{N^-}$ be respectively ascending and descending ladder heights, and

$$F^+(x) = \mathbf{P}(Z_+ < x),\ F^-(x) = \mathbf{P}(Z_- > x).$$

Denote, by $H_+(x)$, the renewal function corresponding to the distribution $F^+$ of the ascending ladder height,

$$H_+(x) = \sum_{n=0}^{\infty} F_n^+(x),$$

where $F_n^+,\ n \ge 1$, is the $n$-th convolution of $F^+$, $F_0$ is the degenerate distribution concentrated at zero. Similarly, the renewal function $H_-(x)$ is defined by $F^-$.

Notice that

$$H_+(x) = F_0(x) + \sum_{n=1}^{\infty} \mathbf{P}\Big(\min_{0<k\le n-1} S_k > 0,\ 0 < S_n < x\Big),$$

$$H_-(x) = F_0(x) + \sum_{n=1}^{\infty} \mathbf{P}\Big(\max_{0\le k\le n-1} S_k \le 0,\ x < S_n \le 0\Big)$$

(see [1], Ch.12, § 2).

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Hence,

$$\mathbf{P}(Z_+ > x) = -\int_{-\infty}^{0+} \mathbf{P}(X > x - y)dH_-(y) = \int_x^\infty H_-(x-y)dF(y), \tag{1}$$

if $x > 0$, and

$$\mathbf{P}(Z_- > x) = \int_{0+}^{\infty} \mathbf{P}(X < x - y)dH_+(y) = \int_{-\infty}^{x} H_+(x-y)dF(y), \tag{2}$$

if $x < 0$.

The measure

$$\nu(D) = \sum_{n=1}^{\infty} \frac{F_n(D)}{n} \tag{3}$$

is called the harmonic renewal measure.

Since

$$F_n(x+l) - F_n(x) < c(F)\frac{l+1}{\sqrt{n}} \tag{4}$$

(see, e.g., [2]), the measure $\nu$ is $\sigma$-finite. Harmonic renewal measures were studied in [3–7].

For every $x > 0$, put $G^+(x) = \nu((0, x))$. Evidently,

$$G^+(x) = \sum_{n=1}^{\infty} n^{-1}[F_n(x) - F_n(0+)] < \infty.$$

Similarly, define $G^-(x) = \nu((x, 0])$ for $x \le 0$. Harmonic renewal measures are of interest for us, first of all, because

$$\int_0^\infty e^{-sx} dH_+(x) = \exp\Big\{-\int_{0+}^{\infty} e^{-sx} dG^+(x)\Big\} \tag{5}$$

and

$$\int_{-\infty}^{0} e^{sx} dH_-(x) = -\exp\Big\{-\int_{-\infty}^{0+} e^{sx} dG^-(x)\Big\}, \tag{6}$$

which provides the possibility of studying the asymptotic behaviour of $H_\pm(\pm x)$ as $x \to \infty$.

**Proposition.** *Let*

$$a := \mathbf{E}X = 0, \ 0 < \sigma^2 := \mathbf{E}X^2 < \infty. \tag{7}$$

*Then*

$$\lim_{s \downarrow 0} \left( \int_{0-}^{\infty} e^{-sx} dG^+(x) + \ln s \right) = Q - \frac{1}{2} \ln \frac{\sigma^2}{2}, \tag{8}$$

*where*

$$Q = \sum_{n=1}^{\infty} n^{-1} \left[ \mathbf{P}(S_n \ge 0) - \frac{1}{2} \right]$$

*is the Spitzer series, and*

$$\lim_{s \downarrow 0} \left( -\int_{-\infty}^{0+} e^{sx} dG^-(x) + \ln s \right) = -Q - \frac{1}{2} \ln \frac{\sigma^2}{2}. \tag{9}$$

Hence, by using the refinement of Karamata's Tauberian theorem given in [8], we immediately obtain

**Corollary 1.** *If conditions (7) hold, then*

$$\lim_{x\to\pm\infty}\Big(G^{\pm}(x)-\ln|x|\Big)=C_0\pm Q-\frac{1}{2}\ln\frac{\sigma^2}{2}, \tag{10}$$

*where $C_0$ is the Euler constant.*

This result is obtained in [7] by using direct probabilistic arguments. Laplace transforms are used in [3,4], however, in the case where the distribution $F$ is concentrated on a semiaxis or stable. In paper [5], the representation for $\nu([-x,x])$ is obtained under the condition that $\mathbf{E}X=0$, $\mathbf{E}|X|^3<\infty$, and some convolution of $F(x)$ has an absolutely continuous component. In that representation, the Spitzer series $Q$ is absent, which is quite explainable since, by (8) and (9),

$$\int_{-\infty}^{\infty}e^{-h|x|}\nu(dx)=-\Big(\ln s+\ln\frac{\sigma^2}{2}\Big).$$

Instead of Laplace transforms, the generalized Fourier transforms are used in [5].

Combining (5) and (8) and then (5) and (9), we obtain

**Corollary 2.** *If conditions (7) are fulfilled, then*

$$\lim_{s\downarrow 0}s\int_{0-}^{\infty}e^{-sx}dH_+(x)=\frac{\sqrt{2}}{\sigma}e^{Q} \tag{11}$$

*and*

$$\lim_{s\downarrow 0}s\int_{-\infty}^{0+}e^{-sx}dH_-(x)=-\frac{\sqrt{2}}{\sigma}e^{-Q}. \tag{12}$$

The Karamata's Tauberian theorem makes it possible to obtain the asymptotics of $H_+(x)$ for $x\to\infty$, namely,

**Corollary 3.** *If conditions (7) hold, then*

$$\lim_{x\to\pm\infty}|x|^{-1}H_{\pm}(x)=\frac{\sqrt{2}}{\sigma}e^{\pm Q}. \tag{13}$$

It is known that, under conditions (7) and (8), $\mathbf{E}Z_+<\infty$, $\mathbf{E}Z_-<-\infty$ (see [9]). Therefore, by the renewal theorem,

$$\lim_{|x|\to\infty}|x|^{-1}H_{\pm}(x)=\frac{1}{|m_{\pm}|}, \tag{14}$$

where $m_{\pm}=\mathbf{E}Z_{\pm}$. Comparing (13) and (14), we conclude that

$$m_{\pm}=\pm\frac{\sigma}{\sqrt{2}}e^{-Q}. \tag{15}$$

We say that the distribution $F$ has a long right tail if, for any $l>0$,

$$\lim_{x\to\infty}\frac{F(x+l)-F(x)}{F(x)}=0. \tag{16}$$

Respectively $F$ has a long left tail if

$$\lim_{x\to-\infty}\frac{F(x+l)-F(x)}{F(x)}=0. \tag{17}$$

**Theorem 1.** *If conditions (7) and (17) hold, then, for $x \to -\infty$,*

$$\mathbf{P}(Z_- < x) \sim \omega_- \int_{-\infty}^{x} F(y)dy, \tag{18}$$

*where* $\omega_- = \frac{\sqrt{2}}{\sigma} e^{Q}$.

*If conditions (7) and (16) hold, then, for $x \to \infty$,*

$$\mathbf{P}(Z_+ > x) \sim \omega_+ \int_{x}^{\infty} (1 - F(y))dy, \tag{19}$$

*where* $\omega_+ = \frac{\sqrt{2}}{\sigma} e^{-Q}$.

Here and below, $a(x) \sim b(x)$ means that $\lim\limits_{x\to\infty} a(x)/b(x) = 1$. We use $c$, $c(\cdot)$, $c(\cdot,\cdot)$ to denote constants which may be different in different contexts.

An asymptotics of large deviation probabilities is studied in [10–13]. The formula

$$\mathbf{P}(Z_+ > x) \sim \frac{1}{m_-} \int_{x}^{\infty} (1 - F(y))dy \tag{20}$$

is obtained, in particular, in [13], and is valid if

$$A := \sum_{n=1}^{\infty} \frac{1}{n} \mathbf{P}(S_n \le x) = \infty, \ \mathbf{E}|Z_-| < \infty.$$

As we have already noticed above, $\mathbf{E}|Z_-| < \infty$ under conditions (7) and (8). In addition, $A = \infty$ in this case. On the other hand, as it is shown above (see (15)), $m_- = \frac{\sigma}{\sqrt{2}} e^{-Q}$. The additional information which is contained in (19) as compared with (20) consists namely in this fact. Notice also that (19) is deduced by the quite different method by comparison with (20).

**Theorem 2.** *Let, for $x \to \infty$,*

$$F(-x) \sim \frac{q}{x^{\alpha}}, \ 1 - F(x) \sim \frac{p}{x^{\alpha}}, \tag{21}$$

*where $0 < \alpha < 1$, $p \ge 0$, $q \ge 0$. Then there exists the slowly varying function $L(x), x > 0$, such that, for $x \to -\infty$,*

$$H_-(x) \sim |x|^{\gamma} L(|x|), \tag{22}$$

*where*

$$\gamma = \frac{\alpha}{2} - \frac{c(\alpha,\beta)}{\pi}, \ c(\alpha,\beta) = \arctan\Bigl(\beta \tan\frac{\pi\alpha}{2}\Bigr), \ \beta = \frac{p-q}{p+q}.$$

The analogous result takes place for $H_+(x)$.

We see that the greatest value $\gamma = \alpha$ and the least $\gamma = 0$ are achieved, respectively, for $\beta = -1$ and for $\beta = 1$. These values of $\beta$ correspond to the extreme types of stable laws with the exponent $\alpha$ which $F$ is attracted to. If $\beta = 0$, then evidently $\gamma = \frac{\alpha}{2}$. Letting $\alpha = 2$ in the last equality, we obtain the value 1 for $\gamma$.

It is not improbable that the function $L(x)$ in Theorem 2 is in fact constant. This is the case if the distribution $F$ is concentrated on the negative semiaxis (see [3]).

The analysis of the proof of Theorem 2 shows that $L(x) = const$ in the case of symmetric $F$.

**Theorem 3.** *If conditions (16) and (21) are fulfilled, then there exists the slowly varying function $l(x)$ such that, for $x \to \infty$,*

$$1 - F^+(x) \sim x^{\gamma-\alpha} l(x), \tag{23}$$

*where*

$$\gamma = \frac{\alpha}{2} - \frac{c(\alpha,\beta)}{\pi}, c(\alpha,\beta) = \arctan\Bigl(\beta \tan\frac{\pi\alpha}{2}\Bigr).$$

Notice that $L(x)$ and $l(x)$, generally speaking, do not satisfy the condition $L(x) \sim cl(x)$. However, if $L(x)$ equals a constant, then it is true for $l(x)$ as well. The result similar to Theorem 3 is also valid for $F^-(x)$.

## 2. Proof of Proposition

Denote, by $f(t)$, the characteristic function of the random variable $X$. The starting point is the next formula deduced in [14] (see also [15])

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{0+}^{\infty} e^{-hx} dF_n(x) + \frac{1}{2} \sum_{n=1}^{\infty} (F_n(0+) - F_n(0)) n^{-1}$$
$$= -\frac{h}{\pi} \int_0^{\infty} \frac{\ln|1-f(t)|}{h^2+t^2} dt - \frac{1}{\pi} \int_0^{\infty} \frac{t \arg(1-f(t))}{h^2+t^2} dt \tag{24}$$

with

$$\frac{h}{\pi} \int_0^{\infty} \frac{\ln|1-f(t)|}{h^2+t^2} dt = -\frac{1}{2} \int_{|x|\neq 0} e^{-h|x|} dG(x), \tag{25}$$

$$\frac{1}{\pi} \int_0^{\infty} \frac{t \arg(1-f(t))}{h^2+t^2} dt = \frac{1}{2} \int_{x<0} e^{hx} dG(x) - \frac{1}{2} \int_{x>0} e^{-hx} dG(x). \tag{26}$$

First, we show that the right-hand side of equality (26) goes to

$$\frac{1}{2} \sum_{n=1}^{\infty} (\mathbf{P}(S_n<0) - \mathbf{P}(S_n>0))$$

as $h \downarrow 0$.

We need several lemmas to proof it.

**Lemma 2.1.** *If $a=0$ and $\sigma^2<\infty$, then*

$$\mathbf{P}(S_n>0) - \mathbf{E}\Big\{e^{-hS_n}; S_n>0\Big\} \le h\sigma\sqrt{n}, \tag{27}$$

*and*

$$\mathbf{P}(S_n<0) - \mathbf{E}\Big\{e^{hS_n}; S_n<0\Big\} < h\sigma\sqrt{n}. \tag{28}$$

*Proof.* Applying the inequalities $1-e^{-x}<x,\ x>0$, and $\mathbf{E}|S_n| \le \sigma\sqrt{n}$, we have

$$\mathbf{P}(S_n>0) - \mathbf{E}\Big\{e^{-hS_n}; S_n>0\Big\} = \mathbf{E}\Big\{1-e^{-hS_n}; S_n>0\Big\} < h\mathbf{E}\Big\{S_n; S_n>0\Big\} < h\sigma\sqrt{n}.$$

In just the same way, (28) is proved. □

Put

$$\Delta_n(h) = \int_{x>0} e^{-hx} dF_n(x) - \int_{x<0} e^{hx} dF_n(x) + \mathbf{P}(S_n<0) - \mathbf{P}(S_n>0). \tag{29}$$

**Lemma 2.2.** *Under conditions of Lemma 2.1,*

$$\lim_{n\to\infty} \sup_{h>0} \big|\Delta_n(h)\big| = 0. \tag{30}$$

*Proof.* Integrating by parts on the right-hand side of formula (29), we find that

$$\Delta_n(h) = h \int_0^{\infty} (1-F_n(x)-F_n(-x)) e^{-hx} dx.$$

Consequently,

$$|\Delta_n(h)| < h \sup_x \Big|1-F_n(x)-F_n(-x)\Big| \int_0^{\infty} e^{-hx} dx.$$

By CLT,

$$\lim_{n\to\infty} \Big(1-F_n(x)-F_n(-x)\Big) = 0.$$

The assertion of the lemma follows from two previous relations. □

**Lemma 2.3.** *If the distribution $F$ is not degenerate at zero, then*

$$\mathbf{E}\Big\{e^{-hS_n}; S_n > 0\Big\} < \frac{c(F)}{h\sqrt{n}}, \tag{31}$$

$$\mathbf{E}\Big\{e^{-hS_n}; S_n < 0\Big\} < \frac{c(F)}{h\sqrt{n}}. \tag{32}$$

*Proof.* We restrict ourselves to proving (31). Obviously,

$$\int_{0+}^{\infty} e^{-hx} dF_n(x) < \sum_{k=0}^{\infty} e^{-hk} \mathbf{P}(k < S_n \le k+1).$$

Since, by (4),

$$\mathbf{P}(k < S_n \le k+1) < \frac{c(F)}{\sqrt{n}},$$

$$\int_{0+}^{\infty} e^{-hx} dF_n(x) < \frac{c(F)}{\sqrt{n}} \sum_{k=0}^{\infty} e^{-hk}.$$

On the other hand,

$$\sum_{k=0}^{\infty} e^{-hk} = \frac{1}{1-e^{-h}} < \frac{1}{h}.$$

The desired result follows from these two inequalities. □

Consider the series

$$\Sigma(h) = \sum_{n=1}^{\infty} n^{-1} \Big|\Delta_n(h)\Big|$$

$$= \sum_{n<\frac{\varepsilon^2}{h^2}} n^{-1}\Big|\Delta_n(h)\Big| + \sum_{\frac{\varepsilon^2}{h^2}<n\le\frac{1}{h^2\varepsilon^2}} n^{-1}\Big|\Delta_n(h)\Big| + \sum_{n>\frac{1}{h^2\varepsilon^2}}^{n-1} n^{-1}\Big|\Delta_n(h)\Big| = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 . \tag{33}$$

Applying Lemma 2.1, we find that

$$\sum\nolimits_1 < h \sum_{n\le\frac{\varepsilon^2}{h^2}} \frac{1}{\sqrt{n}} < 3\varepsilon. \tag{34}$$

Further,

$$\sum\nolimits_2 < \sup_{\frac{\varepsilon^2}{h^2}<n\le\frac{1}{h^2\varepsilon^2}} \Delta_n(h)\left(\frac{h^2}{\varepsilon^2} + \int_{\frac{\varepsilon^2}{h^2}}^{\frac{1}{\varepsilon^2 h^2}} \frac{dx}{x}\right).$$

Since

$$\int_{\frac{\varepsilon^2}{h^2}}^{\frac{1}{\varepsilon^2 h^2}} \frac{dx}{x} = -4\ln\varepsilon,$$

we have, by (30),

$$\lim_{h\downarrow 0} \sum\nolimits_2 = 0. \tag{35}$$

Notice that, by Lemma 2.3,

$$\Big|\Delta_n(h)\Big| < \frac{c(F)}{h\sqrt{n}} + |2F_n(0)-1|.$$

Therefore,

$$\sum\nolimits_3 < \frac{c(F)}{h} \sum_{n>\frac{1}{\varepsilon^2 h^2}} \frac{1}{n^{3/2}} + \sum_{n>\frac{1}{(\varepsilon h)^2}} n^{-1}|2F_n(0) - 1|.$$

Further,

$$\sum_{n>\frac{1}{(\varepsilon h)^2}} \frac{1}{n^{3/2}} < h^3\varepsilon^3 + \int_{u>(\varepsilon h)^{-2}}^{\infty} \frac{du}{u^{3/2}} < h^3\varepsilon^3 + 2\varepsilon h.$$

The series

$$\sum_1^\infty n^{-1}(2F_n(0) - 1)$$

absolutely converges (see [16]). Thus,

$$\overline{\lim_{h\downarrow 0}} \sum\nolimits_3 < 2\varepsilon. \tag{36}$$

It follows from (33) - (36) that

$$\overline{\lim_{h\downarrow 0}}\, \Sigma(h) < 5\varepsilon.$$

It means by (29) that

$$\lim_{h\downarrow 0}\Big(\int_{x>0} e^{-hx} dG(x) - \int_{x<0} e^{hx} dG(x)\Big) = \sum_{n=1}^{\infty} n^{-1}\Big(\mathbf{P}(S_n > 0) - \mathbf{P}(S_n < 0)\Big). \tag{37}$$

Proceed now to the left-hand side of equality (25). Choose $\delta$ in the partition

$$\int_0^\infty \frac{\ln|1 - f(t)|}{t^2 + h^2} dt = \Big(\int_0^\delta + \int_\delta^\infty\Big) \frac{\ln|1 - f(t)|}{t^2 + h^2} dt = I_1(h) + I_2(h) \tag{38}$$

in such a way as the function $f(t) \neq 1$ in the interval $(0, \delta)$. Put

$$f_1(t) = 2\frac{1 - f(t)}{\sigma^2 t^2}.$$

Then

$$I_1(h) = \int_0^\delta \frac{\ln f_1(t)}{t^2 + h^2} dt$$

$$+ \ln\frac{\sigma^2}{2} \int_0^\delta \frac{dt}{t^2 + h^2} + 2\int_0^\delta \frac{\ln t}{t^2 + h^2} dt = I_{11}(h) + \ln\Big(\frac{\sigma^2}{2}\Big) I_{12}(h) + I_{13}(h). \tag{39}$$

Since $f_1(0) = 1$,

$$\lim_{h\downarrow 0} h I_{11}(h) = 0. \tag{40}$$

Further,

$$\lim_{h\downarrow 0} h I_{12}(h) = \frac{\pi}{2}. \tag{41}$$

It is easily seen that

$$\lim_{h\downarrow 0} h \int_\delta^\infty \frac{\ln t}{t^2 + h^2} dt = 0.$$

Consequently, for $h \downarrow 0$,

$$h I_{13}(h) = h \int_0^\infty \frac{\ln t}{t^2 + h^2} dt + o(1).$$

On the other hand,

$$\int_0^\infty \frac{\ln t}{t^2 + h^2} dt = \frac{\ln h}{h} \int_0^\infty \frac{dt}{1 + t^2} + \frac{1}{h} \int_0^\infty \frac{\ln t}{1 + t^2} dt = \frac{\pi}{2h} \ln h. \tag{42}$$

We use the equality

$$\int_0^\infty \frac{\ln t}{1+t^2}dt = 0$$

(see [17], p. 546, Section 4.231, formula 8). Thus, for $h \downarrow 0$,

$$hI_{13}(h) = \frac{\pi}{2}\ln h + o(1). \tag{43}$$

It follows from (39) - (43) that

$$hI_1(h) = \pi\Big(\ln h + \frac{1}{2}\ln\frac{\sigma^2}{2}\Big) + o(1). \tag{44}$$

Estimate now $I_2(h)$. First of all,

$$\Big|\ln|1-f(t)|\Big| < \sum_{n=1}^\infty n^{-1}|f^n(t)|.$$

Consequently,

$$I_2(h) < \sum_{n=1}^\infty \frac{1}{n}\int_\delta^\infty \frac{|f(t)|^n}{t^2}dt.$$

The next bound

$$\int_{|t-v|\le 0.65\frac{\widetilde{\sigma}^2(L)}{\widetilde{\beta}_3(L)}} |f^n(t)|\,dt \le \frac{c}{\widetilde{\sigma}(L)\sqrt{n}} \tag{45}$$

holds, where $\widetilde{\sigma}^2(L) = \int_{|x|\le L} x^2\,d\widetilde{F}(x)$, $\widetilde{\beta}_3(L) = \int_{|x|\le L}|x|^3\,d\widetilde{F}(x)$, $c < 7.61579$, $\widetilde{F}(x)$ is the symmetrization of $X$ (see [14], Lemma 2.1.)

Splitting the interval $(\delta,\infty)$ into intervals of the length $1.3\frac{\widetilde{\sigma}^2(L)}{\widetilde{\beta}_3(L)}$ and applying bound (45), it is not hard to show that

$$\int_\delta^\infty \frac{|f(t)|^n}{t^2}dt < \frac{c(F)}{\sqrt{n}}.$$

Consequently, uniformly in $h > 0$,

$$I_2(h) < c(F). \tag{46}$$

Returning now to (38) and taking (44) and (45) into account, we obtain that, for $h \downarrow 0$

$$\frac{h}{\pi}\int_0^\infty \frac{\ln|1-f(t)|}{t^2+h^2}dt = \ln h + \frac{1}{2}\ln\frac{\sigma^2}{2} + o(1). \tag{47}$$

Combining (24), (37), and (47), we arrive at formula (8). Formula (9) is deduced in the same way. □

## 3. Proof of Theorem 1

Without loss of generality, we may assume that $F(x)$ is continuous. By (13) for any $x < x(\epsilon) < 0$,

$$-(1-\epsilon) < x\frac{\sigma}{\sqrt{2}}H_-(x) < (1+\epsilon)x. \tag{48}$$

Using (1), we have

$$\mathbf{P}(Z_+x) = \int_{x-x(\epsilon)}^\infty H_-(x-y)dF(y) + \int_x^{x-x(\epsilon)} H_-(x-y)dF(y) = I_1(x) + I_2(x). \tag{49}$$

Obviously, by (48),

$$(1-\epsilon)\frac{\sigma}{\sqrt{2}}\int_{x-x(\epsilon)}^\infty (y-x)dF(y) < I_1(x) < \frac{\sigma}{\sqrt{2}}\int_{x-x(\epsilon)}^\infty (y-x)dF(y)(1+\epsilon). \tag{50}$$

Further,

$$I_2(x) < H_-(x(\epsilon))(F(x(\epsilon)) - F(x)).$$

Hence, by condition (16) for $x \to \infty$,

$$I_2(x) = o(1 - F(x)). \tag{51}$$

By the same reason for $x \to \infty$,

$$\int_x^{x-x(\epsilon)} dF(y) = o(1 - F(x)). \tag{52}$$

Put

$$\overline{F}(x) = \int_x^\infty (y - x)dF(y) = \int_x^\infty (1 - F(y))dy.$$

It is easily seen that

$$1 - F(x) = o(\overline{F}(x)). \tag{53}$$

It follows from (51) and (53) that

$$I_2(x) = o(\overline{F}(x)), \tag{54}$$

and from (52), (53)

$$\int_{x-x(\epsilon)}^\infty (y - x)dF(y) \sim \overline{F}(x). \tag{55}$$

By (50) and (55),

$$\frac{\sigma}{\sqrt{2}}(1 - \epsilon) \le \lim_{x\to\infty} \inf I_1(x)/\overline{F}(x) \le \lim_{x\to\infty} \sup I_2(x)/\overline{F}(x) \le \frac{\sigma}{\sqrt{2}}(1 + \epsilon). \tag{56}$$

Combining (49), (54), and (56), we get the desired result. □

## 4. Proof of Theorem 2

Previously, we prove several lemmas.

**Lemma 4.1.** *Let, for $x \to \infty$, $1 - F(x) \sim \frac{c}{x^\alpha}$, $0 < \alpha < 1$. Then for $t \downarrow 0$*

$$\int_0^\infty \sin(tx)dF(x) \sim ct^\alpha \int_0^\infty \frac{\cos x}{x^\alpha} dx. \tag{57}$$

*Proof.* Clearly,

$$\int_0^\infty \sin(tx)dF(x) = \int_0^{M/t} \sin(tx)dF(x) + \int_{M/t}^\infty \sin(tx)dF(x). \tag{58}$$

There exists the constant $K$ such that

$$1 - F(x) < \frac{Kc}{x^\alpha}. \tag{59}$$

Therefore, for any $M > 0$,

$$\Big|\int_{M/t}^\infty \sin(tx)dF(x)\Big| < 1 - F(M/t) < \frac{Kc}{M^\alpha}t^\alpha. \tag{60}$$

Integrating by parts, we have

$$\int_0^{M/t} \sin(tx)dF(x) = t\int_0^{M/t} (1 - F(x))\cos(tx)dx + (1 - F(M/t))\sin(M/t).$$

Hence, by (59),

$$\Big|\int_0^{M/t} \sin(tx)dF(x) - t\int_0^{M/t} (1 - F(x))\cos(tx)dx\Big| < \frac{Kc}{M^\alpha}t^\alpha. \tag{61}$$

Further,

$$\int_0^{M/t}(1-F(x))\cos(tx)dx=\int_{\varepsilon/t}^{M/t}(1-F(x))\cos(tx)dx+\int_0^{\varepsilon/t}(1-F(x))\cos(tx)dx.$$

Hence, by (59),

$$\Big|\int_0^{\varepsilon/t}(1-F(x))\cos(tx)dx\Big|<Kc\int_0^{\varepsilon/t}\frac{dx}{x^\alpha}<\frac{Kc\varepsilon^{1-\alpha}}{1-\alpha}t^{\alpha-1}.$$

For $M$ and $\varepsilon$ fixed,

$$\int_{\varepsilon/t}^{M/t}(1-F(x))\cos(tx)dx=\frac{c}{t^{1-\alpha}}\Big(\int_\varepsilon^M\frac{\cos x}{x^\alpha}dx+o(1)\Big).$$

As a result, we obtain, for every fixed $M$ and $\varepsilon$,

$$\int_0^{M/t}(1-F(x))\cos(tx)dx=ct^{\alpha-1}\Big(\int_\varepsilon^M\frac{\cos x}{x^\alpha}dx+o(1)\Big)+\theta\frac{Kc\varepsilon^{1-\alpha}}{1-\alpha}t^{\alpha-1},\quad |\theta|\le 1. \quad (62)$$

Returning now to (61), we conclude that

$$\int_0^{M/t}\sin(tx)dF(x)=ct^\alpha\Big(\int_\varepsilon^M\frac{\cos x}{x^\alpha}dx+o(1)\Big)+\theta KCt^\alpha\Big(M^{-\alpha}+\frac{\varepsilon^{1-\alpha}}{1-\alpha}\Big). \quad (63)$$

The desired result follows from (58), (60), and (62). □

**Lemma 4.2.** *Under conditions of Lemma 4.1 for* $t\to 0$,

$$\int_0^\infty(1-\cos(tx))dF(x)\sim c|t|^\alpha\int_0^\infty\frac{\sin x}{x^\alpha}dx. \quad (64)$$

*Proof.* Without loss of generality, we may assume $t>0$. Obviously,

$$\int_0^\infty(1-\cos(tx))dF(x)=\int_0^{M/t}(1-\cos(tx))dF(x)+\int_{M/t}^\infty(1-\cos(tx))dF(x). \quad (65)$$

By (59) for $M>0$,

$$1-F(M/t)<\frac{Kc}{M^\alpha}t^\alpha. \quad (66)$$

Hence,

$$\int_{M/t}^\infty(1-\cos(tx))dF(x)<\frac{Kc}{M^\alpha}t^\alpha. \quad (67)$$

Integrating by parts, we have

$$\int_0^{M/t}(1-\cos(tx))dF(x)=t\int_0^{M/t}(1-F(x))\sin(tx)dx+(1-F(M/t))(1-\cos M).$$

Hence, by (67),

$$\Big|\int_0^{M/t}(1-\cos(tx))dF(x)-t\int_0^{M/t}(1-F(x))\sin(tx)dx\Big|<\frac{Kc}{M^\alpha}t^\alpha. \quad (68)$$

Further,

$$\int_0^{M/t}(1-F(x))\sin(tx)dx=\int_0^{\varepsilon/t}(1-F(x))\sin(tx)dx+\int_{\varepsilon/t}^{M/t}(1-F(x))\sin(tx)dx.$$

By (59),

$$\Big|\int_0^{\varepsilon/t}(1-F(x))\sin(tx)dx\Big| < \frac{Kc\varepsilon^{1-\alpha}}{1-\alpha}t^{\alpha-1}.$$

For $M$ and $\varepsilon$ fixed,

$$\int_{\varepsilon/t}^{M/t}(1-F(x))\sin(tx)dx = \frac{c}{t^{1-\alpha}}\Big(\int_\varepsilon^M \frac{\sin x}{x^\alpha}dx + o(1)\Big).$$

It follows from two last formulas that

$$t\int_0^{M/t}(1-F(x))\sin(tx)dx = ct^\alpha\Big(\int_\varepsilon^M \frac{\sin x}{x^\alpha}dx + o(1)\Big) + \theta\frac{Kct^\alpha}{1-\alpha}\varepsilon^{1-\alpha}, \quad |\theta| \le 1. \tag{69}$$

Combining (65)–(69), we obtain the assertion of Lemma 4.2. □

**Lemma 4.3.** *For any* $0 < \alpha < 1$,

$$\int_0^\infty \frac{e^{ix}}{x^\alpha}dx = i^{1-\alpha}\Gamma(1-\alpha). \tag{70}$$

*Proof.* By changing the contour of integration in accordance with the change of the variable $x = iy$, we find

$$\int_0^\infty \frac{e^{ix}}{x^\alpha}dx = i^{1-\alpha}\int_0^\infty e^{-y}y^{-\alpha}dy = i^{1-\alpha}\Gamma(1-\alpha). \qquad \square$$

**Lemma 4.4.** *Let* $F(x)$ *satisfy conditions (21). Then, for* $t \to 0$,

$$1 - \int_{-\infty}^\infty e^{itx}dF(x) \sim |t|^\alpha\Big((p+q)\cos\frac{\pi\alpha}{2} + i(q-p)\frac{t}{|t|}\sin\frac{\pi\alpha}{2}\Big)\Gamma(1-\alpha). \tag{71}$$

*Proof.* Obviously,

$$1 - \int_{-\infty}^\infty e^{itx}dF(x) = \int_{-\infty}^\infty (1-\cos(tx))dF(x) - i\int_{-\infty}^\infty \sin(tx)dF(x).$$

By (57) and (64) for $t \to 0$,

$$\int_{-\infty}^\infty (1-\cos(tx))dF(x) \sim (p+q)|t|^\alpha \int_0^\infty \frac{\sin x}{x^\alpha}dx$$

and

$$\int_{-\infty}^\infty \sin(tx)dF(x) \sim (p-q)|t|^\alpha\frac{t}{|t|}\int_0^\infty \frac{\cos x}{x^\alpha}dx.$$

Thus,

$$1 - \int_{-\infty}^\infty e^{itx}dF(x) \sim |t|^\alpha\Big((p+q)\int_0^\infty \frac{\sin x}{x^\alpha}dx + i(q-p)\frac{t}{|t|}\int_0^\infty \frac{\cos x}{x^\alpha}dx\Big).$$

According to Lemma 4.3,

$$\int_0^\infty \frac{\cos x}{x^\alpha}dx = \Gamma(1-\alpha)Re\, i^{1-\alpha} = \sin\frac{\pi\alpha}{2},$$

$$\int_0^\infty \frac{\sin x}{x^\alpha}dx = \Gamma(1-\alpha)Im\, i^{1-\alpha} = \cos\frac{\pi\alpha}{2}.$$

Substituting these values into the previous equality, we obtain the desired result. □

Considering as before that $F(x)$ satisfies conditions (21),we study the Laplace transform of the projection of the harmonic renewal measure (3) on the semiaxis $(-\infty, 0]$.

It is easily seen that

$$\int_{-\infty}^{0-} e^{hx} dG^-(x) = -\int_{0+}^{\infty} e^{-hx} dG^-(-x). \tag{72}$$

Hence,

$$\int_{-\infty}^{0+} e^{-hx} dG^-(x) = \int_{0+}^{\infty} e^{hx} dG^-(x) + \nu(\{0\}). \tag{73}$$

By (24),

$$\int_{0+}^{\infty} e^{-hx} dG^-(-x) + \frac{1}{2}\nu(\{0\}) = -\frac{h}{\pi}\int_0^{\infty} \frac{\ln|1-f(-t)|}{h^2+t^2} dt - \frac{1}{\pi}\int_0^{\infty} \frac{t\arg(1-f(-t))}{h^2+t^2} dt. \tag{74}$$

Using Lemma 4.4, it is easy to verify that, for $t \to 0$,

$$|1-f(-t)| = |t|^{\alpha}\Gamma(1-\alpha)(p^2+q^2-2pq\sin\pi\alpha)^{1/2} + o(1).$$

Consequently, for $t \to 0$,

$$\ln|1-f(-t)| = \alpha\ln|t| + c(\alpha,p,q) + o(1),$$

where

$$c(\alpha,p,q) = \frac{1}{2}\ln(p^2+q^2-2pq\sin\pi\alpha) + \ln\Gamma(1-\alpha).$$

Hence, by (42),

$$\frac{h}{\pi}\int_0^{\infty} \frac{\ln|1-f(-t)|}{h^2+t^2} dt = \frac{\alpha h}{\pi}\int_0^{\infty} \frac{\ln t}{h^2+t^2} dt + \psi(h) = \frac{\alpha}{2}\ln h + \psi(h), \tag{75}$$

where $\lim_{h\to 0}\psi(h) = c(\alpha,p,q)$.

According to (71),

$$\arg(1-f(-t)) = \frac{t}{|t|}\arctan\Big(\beta\,\mathrm{tg}\,\frac{\pi\alpha}{2}\Big) + \varphi(t), \tag{76}$$

where $\beta = (p-q)/(p+q)$, $\varphi(t) \to 0$ as $t \to 0$.

**Lemma 4.5.** *The function*

$$W(h) = \exp\Big\{\int_h^1 \frac{t\varphi(t)}{h^2+t^2} dt\Big\}, \tag{77}$$

*where $\varphi(t) \to 0$ for $t \to 0$, is slowly varying as $h \downarrow 0$, i.e. for any $c > 0$,*

$$\lim_{h\downarrow 0}\frac{W(ch)}{W(h)} = 1.$$

*Proof.* Changing the variable $t = u^{-1}$, we have

$$I(h) := \int_h^1 \frac{t\varphi(t)}{h^2+t^2} dt = \int_1^{1/h} \frac{\varphi(1/u)}{1+h^2u^2}\frac{du}{u}$$

$$= \int_1^{1/h} \frac{\varphi(1/u)}{u} du - h^2\int_1^{1/h} \frac{u\varphi(1/u)}{1+h^2u^2} du = I_1(h) + I_2(h).$$

It is easily seen that

$$\Big|\int_1^{1/h} \frac{u\varphi(1/u)}{1+h^2u^2} du\Big| < \int_1^{1/h} |\varphi(1/u)| u\, du = o(h^{-2}).$$

Consequently,

$$\lim_{h\to 0} I_2(h) = 0.$$

According to Karamata's criterion, the function

$$Z(x) = \exp\Big\{\int_1^x \frac{\varphi(1/u)}{u} du\Big\}$$

is slowly varying as $x \to \infty$. Hence, $Z(1/h)$ is slowly varying as $h \downarrow 0$. Since

$$W(h) = Z(1/h)\exp\{I_2(h)\},$$

the function $W(h)$ has the same property as well. □

**Lemma 4.6.** *Let a function $\varphi(t)$ be continuous, and $\varphi(0) = 0$. Then*

$$\lim_{h\downarrow 0} \int_{-h}^{h} \frac{t\varphi(t)}{t^2+h^2} dt = 0.$$

*Proof.* The conclusion of the lemma follows from the inequalities

$$\Big|\int_{-h}^{h} \frac{t\varphi(t)}{t^2+h^2} dt\Big| < 2 \sup_{|t|\le h} |\varphi(t)| \int_0^h \frac{t}{t^2+h^2} dt < \sup_{|t|\le h} |\varphi(t)|.$$ □

**Lemma 4.7.** *For $h \downarrow 0$,*

$$\int_0^1 \frac{t\arg(1-f(-t))}{t^2+h^2} dt = c(\alpha,\beta)\ln\frac{1}{h} + \ln W(h) + o(1) \tag{78}$$

*Proof.* Based on formula (76) and Lemmas 4.5 and 4.6, we can state that

$$\int_0^1 \frac{t\arg\,(1-f(-t))}{t^2+h^2} dt = c(\alpha,\beta)\int_0^1 \frac{t}{t^2+h^2} dt + \ln W(h) + o(1),$$

where

$$c(\alpha,\beta) = \arctan\,(\beta\tan\frac{\pi\alpha}{2}).$$

Obviously,

$$\int_0^1 \frac{t}{t^2+h^2} dt = \frac{1}{2}\ln(t^2+h^2)\Big|_0^1 = \ln\frac{1}{h} + \frac{1}{2}\ln(1+h^2).$$

The conclusion of the lemma follows from last two formulas. □

Consider the integral

$$I(h) = \frac{1}{\pi}\int_1^\infty \frac{t\;\arg\,(1-f(-t))}{t^2+h^2} dt.$$

By (24) and (78), $I(h) < \infty$ for $h > 0$.

**Lemma 4.8.** *For every distribution $F$, there exists the finite limit*

$$\lim_{h\to 0} I(h) = I_0.$$

*Proof.* Arguing in the same way as in deducing formula (2.20) in [14], we are sure that

$$\frac{1}{\pi}\int_1^\infty \frac{t(1-\arg f(-t))}{t^2+h^2} dt = -\sum_{n-1}^{\infty} n^{-1}\int_1^\infty \frac{t\mathrm{Im}\, f^n(-t)}{t^2+h^2} dt. \tag{79}$$

Evidently,

$$\int_1^\infty \frac{t\mathrm{Im}\, f^n(-t)}{t^2+h^2} dt = -\int_1^\infty \frac{t\mathrm{Im}\, f^n(t)}{t^2+h^2} dt.$$

Further,

$$\int_1^\infty \frac{t\mathrm{Im}\, f^n(t)}{t^2+h^2} dt = \int_1^\infty \frac{t}{t^2+h^2}\int_{-\infty}^{\infty} \sin(tx) dF_n(x) = \int_{-\infty}^{\infty} dF_n(x)\int_1^\infty \frac{t\sin\,(tx)}{t^2+h^2} dx.$$

By Lemma 1.1 in [14] for $\alpha < h \le 1$,

$$\Big|\int_1^\infty \frac{t\sin(tx)}{t^2+h^2}dt\Big| < \begin{cases} 2/|x|, & |x| \ge 1, \\ 3, & |x| < 1. \end{cases}$$

It follows from last two relations that

$$\Big|\int_1^\infty \frac{t\mathrm{Im}\, f^n(t)}{t^2+h^2}dt\Big| < 3\int_{|x|<n^{1/4}} dF_n(x) + 2\int_{|x|>n^{1/4}} \frac{dF_n(x)}{|x|} < \frac{c(F)}{n^{1/4}}.$$

We have applied here a bound for the concentration function (4). Thus, series (79) converges uniformly in the interval $(0,1]$.

On the other hand, for any $n$,

$$\lim_{h\downarrow 0}\int_1^\infty \frac{t\mathrm{Im}\, f^n(t)}{t^2+h^2}dt = \int_1^\infty \frac{\mathrm{Im}\, f^n(t)}{t}dt.$$

Consequently,

$$\lim_{h\downarrow 0} I(h) = \frac{1}{\pi}\sum_{n=1}^\infty \int_1^\infty \frac{\mathrm{Im}\, f^n(t)}{t}dt = I_0. \qquad \square$$

It follows from Lemmas 4.7 and 4.8 that, for $h \downarrow 0$,

$$\int_0^\infty \frac{t\arg(1-f(-t))}{t^2+h^2}\,dt = c(\alpha,\beta)\ln\frac{1}{h} + \ln W(h) + I_0 + o(1). \tag{80}$$

Combining (72)–(75) and (80), we conclude that

$$\begin{aligned} -\int_{-\infty}^{0+} e^{hx}dG^-(x) &= \int_{0+}^\infty e^{-hx}dG^-(-x) + \nu(\{0\}) \\ &= \gamma\ln\frac{1}{h} - \pi^{-1}(\ln W(h) + c_0(F)) + \frac{1}{2}\nu(\{0\}), \end{aligned}$$

where

$$c_0(F) = c(\alpha,p,q) + I_0,\ \ \gamma = \frac{\alpha}{2} - \frac{c(\alpha,\beta)}{\pi}.$$

Applying now the Baxter identity (see, e.g., [1], Ch. 18, § 3), we find that

$$\begin{aligned} \int_{0+}^\infty e^{-hx}dH_-(-x) &= \exp\Big\{\int_{0+}^\infty e^{-hx}dG^-(-x)\Big\} \\ &\sim h^{-\gamma}\exp\Big\{-\frac{1}{\pi}\Big(\ln W(h) + c_0(F)\Big) + \frac{1}{2}\nu(\{0\})\Big\}. \end{aligned} \tag{81}$$

Using the Tauberian theorem for the Laplace transform (see [1], Ch. 13, § 5), we have

$$H_-(-x) \sim x^\gamma L(x), \tag{82}$$

where

$$L(x) = \frac{1}{\Gamma(1+\gamma)}\exp\Big\{-\frac{1}{\pi}\Big(\ln W(x^{-1}) + c_0(F)\Big) + \frac{1}{2}\nu(\{0\})\Big\},\ \ x>0,$$

which is equivalent to the assertion of the theorem. $\square$

## 5. Proof of Theorem 3

Using formula (1), we have

$$\mathbf{P}(Z_+ > x) \sim p\int_{-\infty}^{0+}(x-y)^{-\alpha}dH_-(y) = p\alpha\int_{-\infty}^{0}(x-y)^{-\alpha-1}H_-(y)dy. \tag{83}$$

We need several lemmas to find the asymptotics of the last integral.

**Lemma 5.1.** *For any $x>0$,*

$$\int_{-\sqrt{x}}^{0}(x-y)^{-\alpha-1}H_-(y)dy < c(\epsilon)x^{\gamma/2-\alpha+\epsilon}, \tag{84}$$

*where $\epsilon$ is as small as one likes.*

*Proof.* By (82), there exists a constant $c$ such that, for every $x>0$,

$$\int_{-\sqrt{x}}^{0}(x-y)^{-\alpha-1}H_-(y)dy < c\int_{-\sqrt{x}}^{0}(x+|y|)^{-\alpha-1}|y|^{\gamma}L(|y|)dy$$

$$= cx^{\gamma-\alpha}\int_{-\frac{1}{\sqrt{x}}}^{0}(1+|y|)^{-\alpha-1}|y|^{\gamma}L(x|y|)dy < c(\epsilon)x^{\gamma/2-\alpha+\epsilon/2} \tag{85}$$

since

$$L(x|y|) < c(\epsilon)(x|y|)^{\epsilon}. \qquad \square$$

**Lemma 5.2.** *As $x\to\infty$,*

$$\Gamma(1+\gamma)\int_{-\infty}^{-\sqrt{x}}(x-y)^{-\alpha-1}H_-(y)dy \sim x^{\gamma-\alpha-1}\int_{-\infty}^{-\sqrt{x}}(1+|y|)^{-\alpha-1}|y|^{\gamma}L(x|y|)dy. \tag{86}$$

*Proof.* The assertion of the lemma follows from asymptotics (82). $\square$

**Lemma 5.3.** *As $x\to\infty$,*

$$\Gamma(1+\gamma)\int_{-\infty}^{0}(x-y)^{-\alpha-1}H_-(y)dy \sim x^{\gamma-\alpha}\int_{-\infty}^{0}(1+|y|)^{-\alpha-1}|y|^{\gamma}L(|y|)dy. \tag{87}$$

*Proof.* Obviously,

$$\int_{-\infty}^{-\sqrt{x}}(x-y)^{-\alpha-1}H_-(y)dy = \int_{-\infty}^{-\sqrt{x}} + \int_{-\sqrt{x}}^{0} = I_1 + I_2.$$

By Lemma 5.2,

$$I_1 > c(\epsilon)x^{\gamma-\alpha-\epsilon.} \tag{88}$$

It follows from (84) and (88) that

$$I_2 = o(I_1).$$

Hence, by (86),

$$\int_{-\infty}^{0}(x-y)^{-\alpha-1}H_-(y)dy$$

$$\sim \int_{-\infty}^{-\sqrt{x}}(x-y)^{-\alpha-1}H_-(y)dy \sim \frac{x^{\gamma-\alpha}}{\Gamma(1+\gamma)}\int_{-\infty}^{-\sqrt{x}}(1+|y|)^{-\alpha-1}|y|^{\gamma}L(x|y|)dy.$$

It remains to remark that, by (85) and (87),

$$\int_{-\infty}^{-\sqrt{x}}(1+|y|)^{-\alpha-1}|y|^{\gamma}L(x|y|)dy \sim \int_{-\infty}^{0}(1+|y|)^{-\alpha-1}|y|^{\gamma}L(x|y|)dy. \quad \square \tag{89}$$

**Lemma 5.4.** *The function*

$$h(x) := \int_{-\infty}^{0} (1+|y|)^{-\alpha-1}|y|^{\gamma}L(x|y|)dy \tag{90}$$

*is slowly varying.*

*Proof.* By (89) for $x \to \infty$,

$$\frac{h(cx)}{h(x)} \sim \frac{\int_{-\infty}^{-\sqrt{x}}(1+|y|)^{-\alpha-1}|y|^{\gamma}L(cx|y|)dy}{\int_{-\infty}^{-\sqrt{x}}(1+|y|)^{-\alpha-1}|y|^{\gamma}L(x|y|)dy} \sim 1. \qquad \square$$

It follows from Lemmas 5.3 and 5.4 that

$$\int_{-\infty}^{0} (x-y)^{-\alpha-1}H_{-}(y)dy \sim \frac{x^{\gamma-\alpha}}{\Gamma(1+\gamma)}h(x), \tag{91}$$

where $h(x)$ is a slowly varying function. Comparing (83) and (91), we find that

$$\mathbf{P}(Z_{+} > x) \sim \frac{p\alpha}{\Gamma(1+\gamma)}x^{\gamma}h(x).$$

Hence, letting

$$l(x) = \frac{p\alpha}{\Gamma(1+\gamma)},$$

we obtain the conclusion of Theorem 3. $\square$

In conclusion, we remark that if the integral

$$\int_0^1 \frac{\varphi(t)}{t}dt,$$

where $\varphi$ is defined by (76), is finite, then $W(h)$ in (77) converges to some constant as $h \downarrow 0$.

Indeed, in this case for any $\eta > h$,

$$\left| \int_h^{\eta} \frac{t\varphi(t)}{t^2+h^2}dt - \int_h^{\eta} \frac{\varphi(t)}{t}dt \right| \le \sup_{0\le t\le \eta} |\varphi(t)| \int_h^{\eta} \left( \frac{t}{t^2+h^2} - \frac{1}{t} \right) dt.$$

Obviously,

$$\int_h^{\eta} \left( \frac{t}{t^2+h^2} - \frac{1}{t} \right) dt < h^2 \int_h^{\eta} \frac{dt}{t^3} < \frac{1}{3}.$$

On the other hand, for every $0 < \eta < 1$,

$$\lim_{h\downarrow 0} \int_{\eta}^{1} \frac{t\varphi(t)}{t^2+h^2}dt = \int_{\eta}^{1} \frac{\varphi(t)}{t}dt.$$

Thus,

$$\lim_{h\downarrow 0} \int_{h}^{1} \frac{t\varphi(t)}{t^2+h^2}dt = \int_{0}^{1} \frac{\varphi(t)}{t}dt.$$

Further, if there exists the finite limit $\lim_{h\downarrow 0} W(h)$, then the same is true for $L(x)$ in (82) as $x \to \infty$. But then, by (89) and (90),

$$h(x) \sim c(\alpha,\gamma)L(x),$$

where

$$c(\alpha,\gamma) = \int_0^{\infty} (1+y)^{-\alpha-1}y^{\gamma}dy.$$

## Bibliography

1. W.Feller, *An Introduction to Probability Theory and Its Applications II*, Wiley, New York, 1967.
2. S.V.Nagaev, S.S.Khodzhabagjan, *On an estimate of the concentration function of sums of independent random variables*, Theory Probab. Appl. **41** (1996), no. 3, 560-568.
3. P.Greenwood, I.Omey and J.L.Teugels, *Harmonic renewal measures*, Z. Warsch. Verw. Gebiete **59** (1982), 391-409.
4. P.Greenwood, I.Omey and J.L.Teugels, *Harmonic renewal measures and bivariate domains of attractions in fluctuation theory*, Z. Warsch. Verw. Gebiete **61** (1982), 527-539.
5. R.J.Grubel, *On harmonic renewal measures*, Probab. Theory Rel. Fields **71** (1986), 393-403.
6. R.J.Grubel, *Harmonic renewal sequences and the first positive sum*, J. London Math. Soc. **38** (1988), no. 2, 179-192.
7. A.J.Stam, *Some theorems on harmonic renewal measures*, Stochastic processes and their Appl. **39** (1991), 277-285.
8. L.de Haan, *An Abel–Tauber theorem for Laplace transforms*, J. London Math. Soc. **13** (1976), no. 3, 537-542.
9. R.A.Doney, *Moments of ladder heights in random walks*, J. Appl. Probab. **17** (1980), 248-252.
10. B.A.Rogosin, *On the distribution of the first jump*, Theory Probab. Appl. **9** (1964), 450-465.
11. N.Veraverbeke, *Asymptotic behaviour of Wiener-Hopf factors of a random walk*, Stochastic processes and their Appl. **5** (1977), 27-37.
12. P.Embrechts, C.M.Goldie, and N.Vereverbeke, *Subexponentiality and infinite divisibility*, Z. Warsch. Verw. Gebiete **49** (1979), 335-347.
13. R.Grubel, *Tail behaviour of ladder–height distibulions in random walks*, J. Appl. Probab. **22** (1985), 705-709.
14. S.V. Nagaev, *Exact expressions for moments of ladder heigsts*, Preprint 2007/192. IM SO RAN, Novosibirsk, 2007 (In Russian).
15. S.V.Nagaev, *The formula for the Laplace transform of the projection of a distribution on the positive semiaxis and some its applications*, Dokl. Math. **416** (2007), no. 5.
16. B.Rosen, *On the asymptotic distribution of sums of independent indentically distributed random variables*, Ark.Mat. **4** (1962), no. 4, 323-332.
17. I.S.Gradshtein, I.M.Ryzhik, *Tables of integrals, sums, series and products*, Fiz. Mat. Giz., Moscow, 1963.

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